Formulação de elementos finitos poligonais

Notas pessoais

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Referências

`PolyTop`: a Matlab implementation of a general topology optimization framework using unstructured polygonal finite element meshes (2012)

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Confirming polygonal finite elements (2004)

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Generalized barycentric coordinates on irregular polygons (2002)

M. Meyer; A. Barr; H. Lee; M. Desbrun

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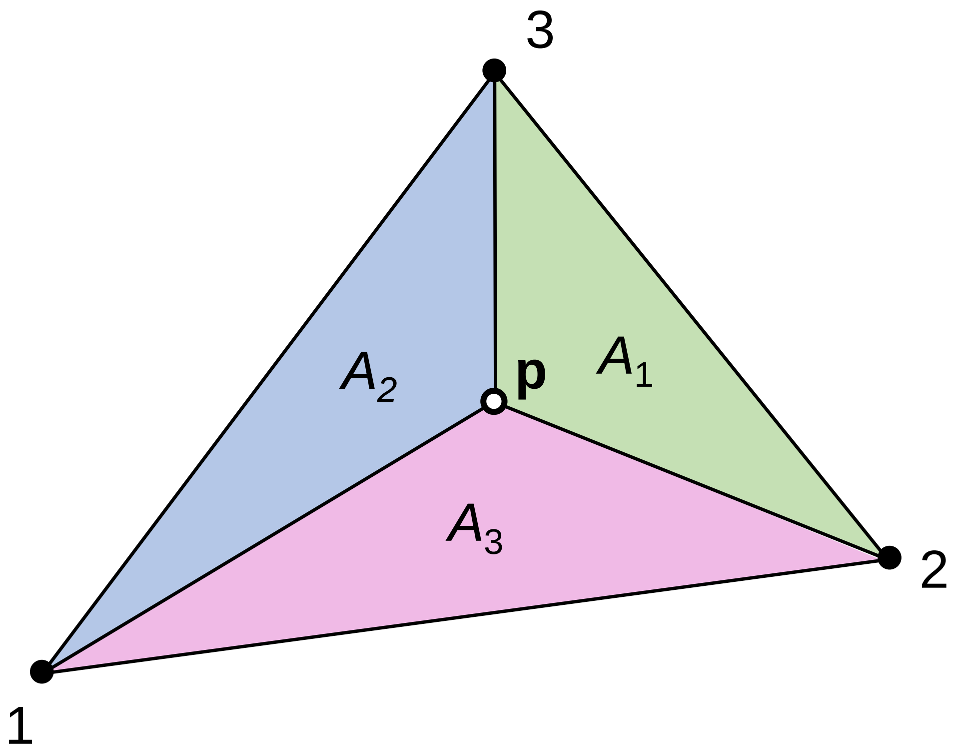
Funções de forma de Wachspress

Coordenadas de área ou baricêntricas

Dado um ponto P no interior de um triângulo de área A, definem-se 3 triângulos internos com áreas A_1, A_2 e A_3:

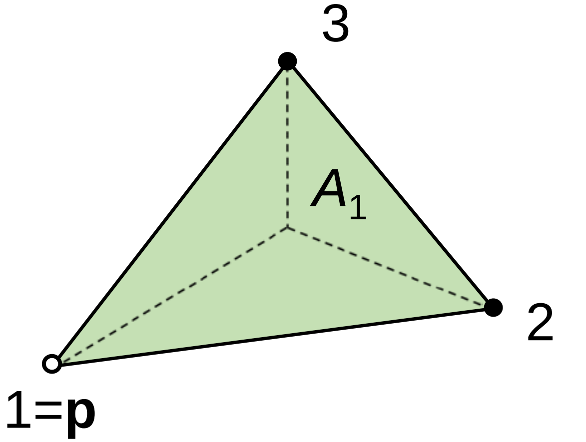


$$A_1 + A_2 + A_3 = A$$

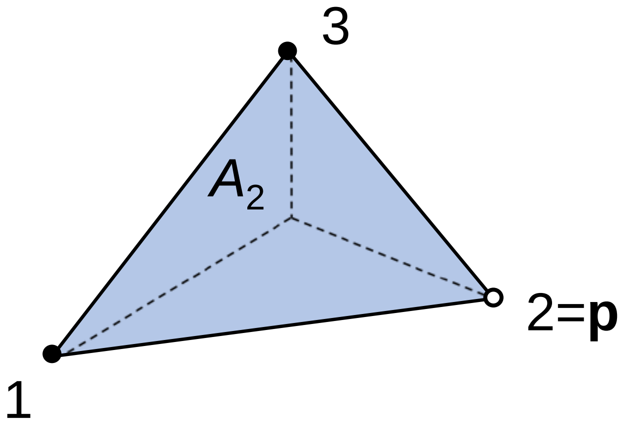
$$\frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = 1 \quad \Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = 1$$

$$\alpha_3 = 1 - \alpha_1 - \alpha_2$$

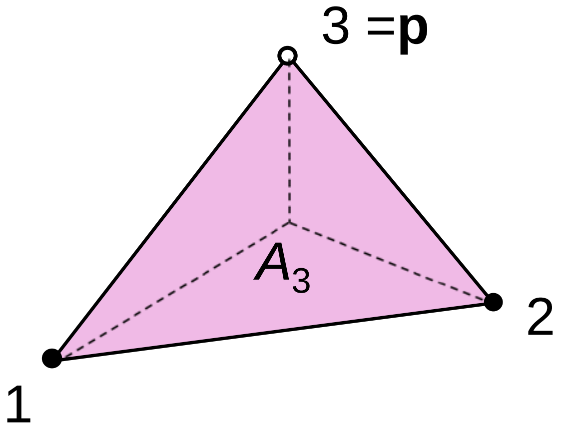
Casos



$$\begin{cases} \alpha_1 = 1 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases}$$

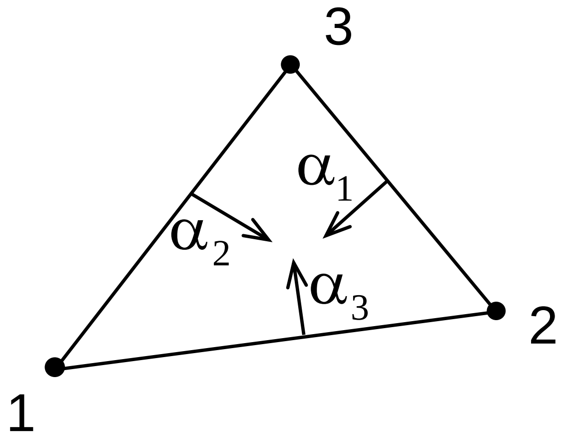


$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 1 \\ \alpha_3 = 0 \end{cases}$$



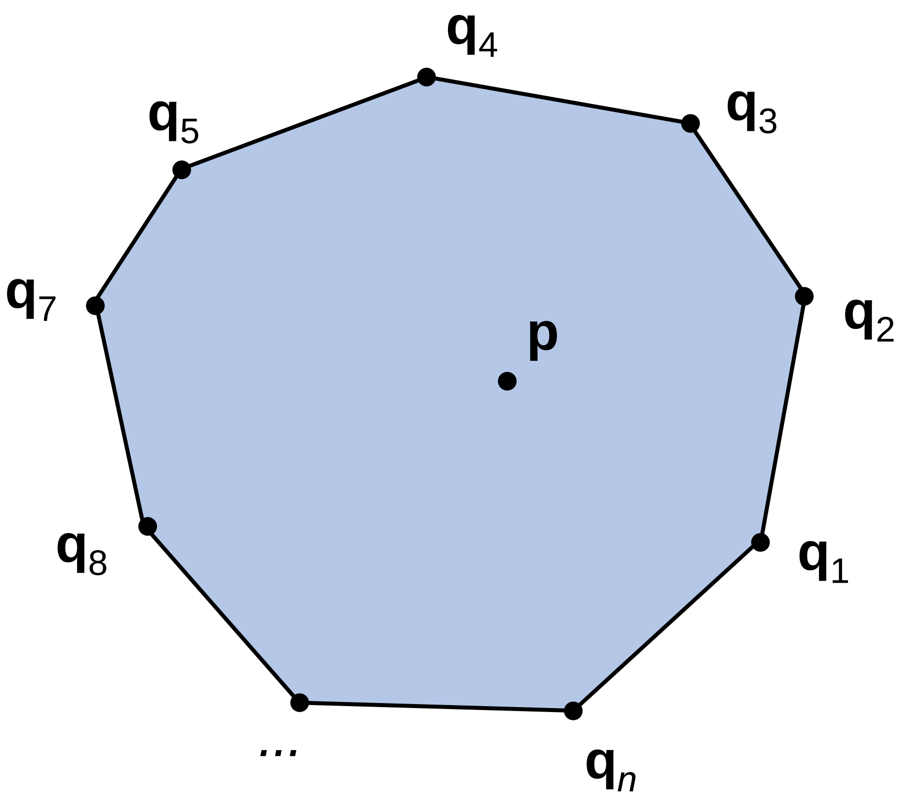
$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 1 \end{cases}$$

As coordenadas baricêntricas são medidas a partir das arestas:



Coordenadas baricêntricas generalizadas

Considere um polígono Q de n vértices: $\mathbf{q}_1$, $\mathbf{q}_2$, ..., $\mathbf{q}_n$, com $n > 2$.



- $\mathbf{p}$ é um ponto arbitrário dentro de Q;
- As coordenadas baricêntricas generalizadas de $\mathbf{p}$ em relação a $\{\mathbf{q}_j\}_{j=1..n}$ são definidas como um conjunto de coeficientes reais (α_1, α_2,..., α_n), dependentes de $\mathbf{p}$ e dos vértices de Q.

Propriedades de $(\alpha_1, \alpha_2, \ldots, \alpha_n)$

1. Combinação afim: esta propriedade permite usar os vértices do polígono como uma base para localizar qualquer ponto em seu interior;

$$\mathbf{p} = \sum_{j=1}^{n} \alpha_j \mathbf{q}_j \qquad\qquad \sum_{j=1}^{n} \alpha_j = 1$$

2. Suavidade: o conjunto deve ser infinitamente diferenciavel em relação a p e aos vértices de *Q*;

3. Combinação convexa: todas as coordenadas são variam entre zero e um:

$$\alpha_j \geq 0, \ \forall j \in [1..n]$$

Coordenadas não-normalizadas

$$\mathbf{p} = \sum_{j=1}^{n} \alpha_j \mathbf{q}_j \quad \Rightarrow \quad \frac{\sum_{j=1}^{n} \alpha_j \mathbf{p}}{\sum_{j=1}^{n} \alpha_j} = \sum_{j=1}^{n} \alpha_j \mathbf{q}_j \quad \Rightarrow \quad \sum_{j=1}^{n} \alpha_j \mathbf{p} = \sum_{j=1}^{n} \alpha_j \mathbf{q}_j$$

$$\sum_{j=1}^{n} \alpha_j = 1$$

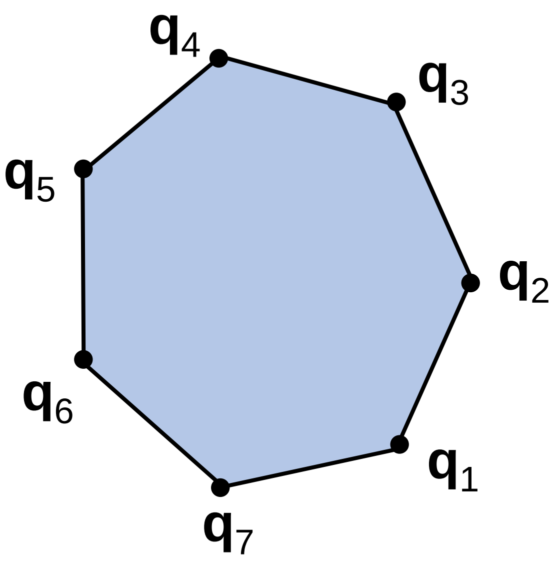
$$\Rightarrow \quad \sum_{j=1}^{n} \alpha_j \left(\mathbf{q}_j - \mathbf{p} \right) = 0$$

$\Rightarrow$ Adotando: $\alpha_j = \omega_j \Big/ \sum_{k=1}^{n} \omega_k$

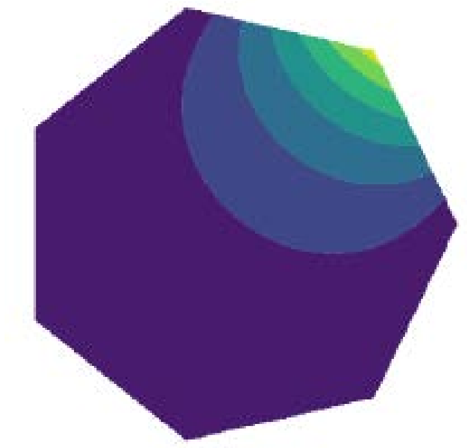
$$\Rightarrow \quad \sum_{j=1}^{n} \omega_j \left(\mathbf{q}_j - \mathbf{p} \right) = 0$$

Proposta:

$$\omega_j = \frac{A\left(\mathbf{q}_{j-1}, \mathbf{q}_j, \mathbf{q}_{j+1}\right)}{A\left(\mathbf{q}_{j-1}, \mathbf{q}_j, \mathbf{p}\right) A\left(\mathbf{q}_j, \mathbf{q}_{j+1}, \mathbf{p}\right)}$$

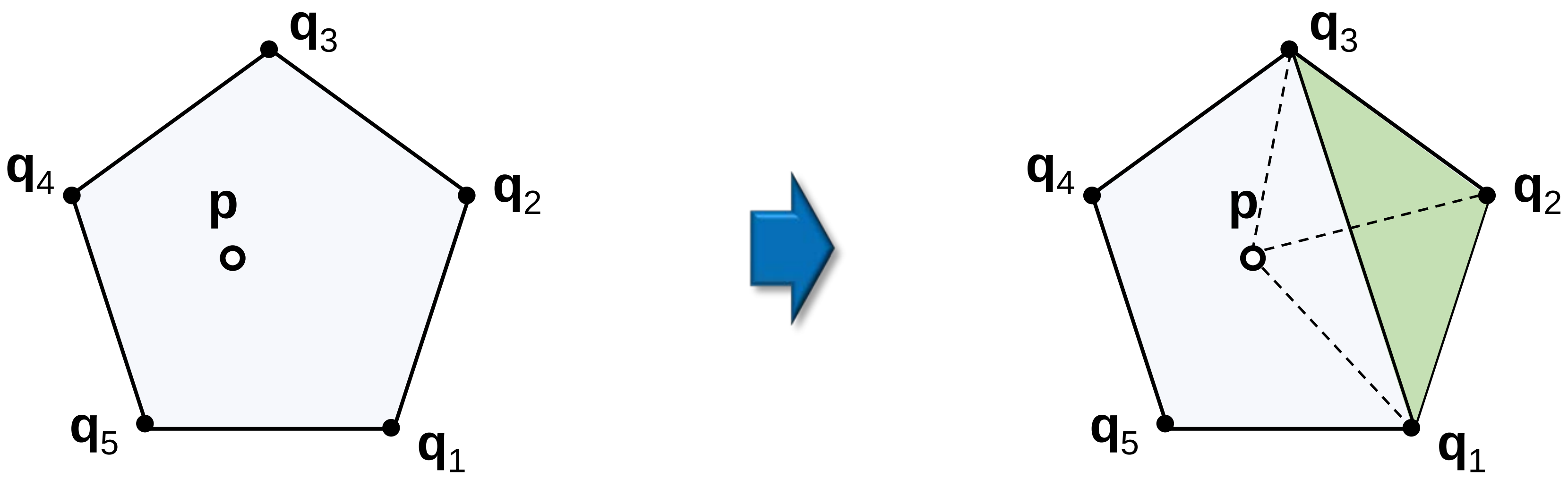


Ex.: Representação ω_3



Dedução
Meyer *et al.* (2002)

Se escrevermos as coordenadas baricêntricas triangulares para o ponto p com respeito ao "triângulo da borda" (destacado):

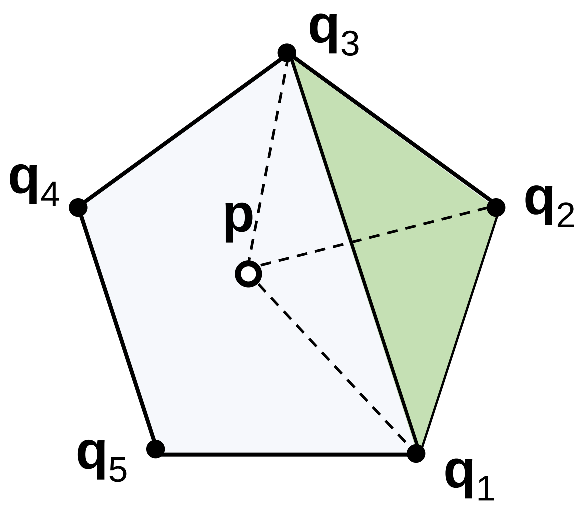


$$\alpha_1 = \frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}$$

$$\alpha_2 = 1 - \frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)} - \frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}$$

$$\alpha_3 = \frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}$$

Dedução
Meyer *et al.* (2002)



$$\mathbf{p} = \sum_{j=1}^{n} \alpha_j \mathbf{q}_j$$

$$\mathbf{p} = \alpha_1\mathbf{q}_1 + \alpha_2\mathbf{q}_2 + \alpha_3\mathbf{q}_3$$

$$\mathbf{p} = \frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}\mathbf{q}_1 + \left(1 - \frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)} - \frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}\right)\mathbf{q}_2 + \frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}\mathbf{q}_3$$

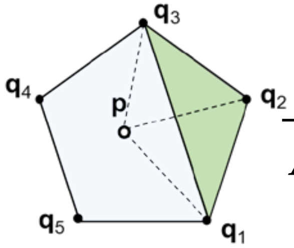
$$A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)(\mathbf{p}-\mathbf{q}_2) = A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})(\mathbf{q}_1-\mathbf{q}_2) + A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})(\mathbf{q}_3-\mathbf{q}_2)$$

$$\frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{p}-\mathbf{q}_2) = \frac{1}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{q}_1-\mathbf{q}_2) + \frac{1}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\mathbf{q}_3-\mathbf{q}_2)$$

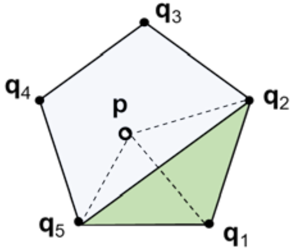
Dedução

Repetindo para os demais "triângulos das bordas":

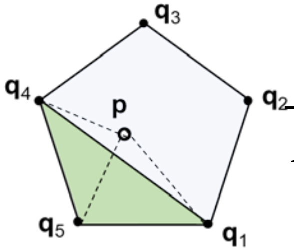
Meyer *et al.* (2002)



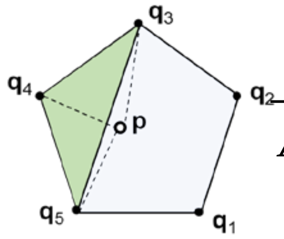
$$\frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{p}-\mathbf{q}_2)=\frac{1}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{q}_1-\mathbf{q}_2)+\frac{1}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\mathbf{q}_3-\mathbf{q}_2)$$



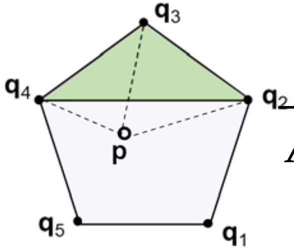
$$\frac{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{q}_2)}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\mathbf{p}-\mathbf{q}_1)=\frac{1}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\mathbf{q}_5-\mathbf{q}_1)+\frac{1}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{q}_2-\mathbf{q}_1)$$



$$\frac{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{q}_1)}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\mathbf{p}-\mathbf{q}_5)=\frac{1}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\mathbf{q}_4-\mathbf{q}_5)+\frac{1}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\mathbf{q}_1-\mathbf{q}_5)$$



$$\frac{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{q}_5)}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\mathbf{p}-\mathbf{q}_4)=\frac{1}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\mathbf{q}_3-\mathbf{q}_4)+\frac{1}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\mathbf{q}_5-\mathbf{q}_4)$$



$$\frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\mathbf{p}-\mathbf{q}_3)=\frac{1}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\mathbf{q}_2-\mathbf{q}_3)+\frac{1}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\mathbf{q}_4-\mathbf{q}_3)$$

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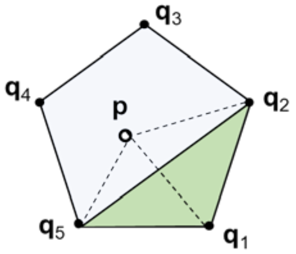
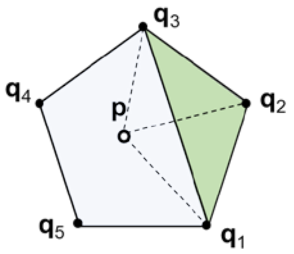
Dedução

Somando as expressões

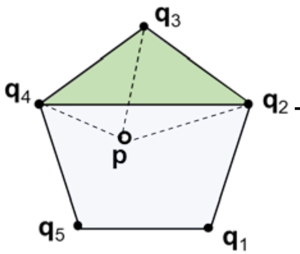
Meyer *et al.* (2002)



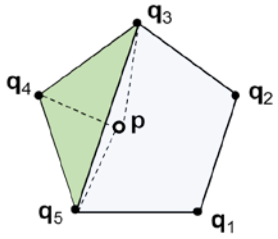
$$\frac{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{q}_2)}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\mathbf{p}-\mathbf{q}_1)=\frac{1}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\cancel{\mathbf{q}_5}-\cancel{\mathbf{q}_1})+\frac{1}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\cancel{\mathbf{q}_2}-\cancel{\mathbf{q}_1})$$



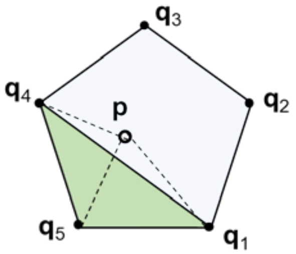
$$\frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{p}-\mathbf{q}_2)=\frac{1}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\cancel{\mathbf{q}_1}-\cancel{\mathbf{q}_2})+\frac{1}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\cancel{\mathbf{q}_3}-\cancel{\mathbf{q}_2})$$



$$\frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\mathbf{p}-\mathbf{q}_3)=\frac{1}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\cancel{\mathbf{q}_2}-\cancel{\mathbf{q}_3})+\frac{1}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\cancel{\mathbf{q}_4}-\cancel{\mathbf{q}_3})$$



$$\frac{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{q}_5)}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\mathbf{p}-\mathbf{q}_4)=\frac{1}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\cancel{\mathbf{q}_3}-\cancel{\mathbf{q}_4})+\frac{1}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\cancel{\mathbf{q}_5}-\cancel{\mathbf{q}_4})$$



$$\frac{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{q}_1)}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\mathbf{p}-\mathbf{q}_5)=\frac{1}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\cancel{\mathbf{q}_4}-\cancel{\mathbf{q}_5})+\frac{1}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\cancel{\mathbf{q}_1}-\cancel{\mathbf{q}_5})$$

Dedução

Somando as expressões

Meyer *et al.* (2002)

Podemos escrever na forma:

$$\frac{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{q}_2)}{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})}(\mathbf{p}-\mathbf{q}_1)+$$

$$\frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p})}(\mathbf{p}-\mathbf{q}_2)+$$

$$\frac{A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)}{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})A(\mathbf{q}_2,\mathbf{q}_3,\mathbf{p})}(\mathbf{p}-\mathbf{q}_3)+$$

$$\frac{A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{q}_5)}{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})A(\mathbf{q}_3,\mathbf{q}_4,\mathbf{p})}(\mathbf{p}-\mathbf{q}_4)+$$

$$\frac{A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{q}_1)}{A(\mathbf{q}_5,\mathbf{q}_1,\mathbf{p})A(\mathbf{q}_4,\mathbf{q}_5,\mathbf{p})}(\mathbf{p}-\mathbf{q}_5)=0$$

$$\Rightarrow \sum_{j=1}^{n=5}\frac{A(\mathbf{q}_{j-1},\mathbf{q}_j,\mathbf{q}_{j+1})}{A(\mathbf{q}_{j-1},\mathbf{q}_j,\mathbf{p})A(\mathbf{q}_j,\mathbf{q}_{j+1},\mathbf{p})}(\mathbf{q}_j-\mathbf{p})=0$$

$$\sum_{j=1}^{n}\omega_j(\mathbf{q}_j-\mathbf{p})=0$$

$$\omega_j=\frac{A(\mathbf{q}_{j-1},\mathbf{q}_j,\mathbf{q}_{j+1})}{A(\mathbf{q}_{j-1},\mathbf{q}_j,\mathbf{p})A(\mathbf{q}_j,\mathbf{q}_{j+1},\mathbf{p})}$$

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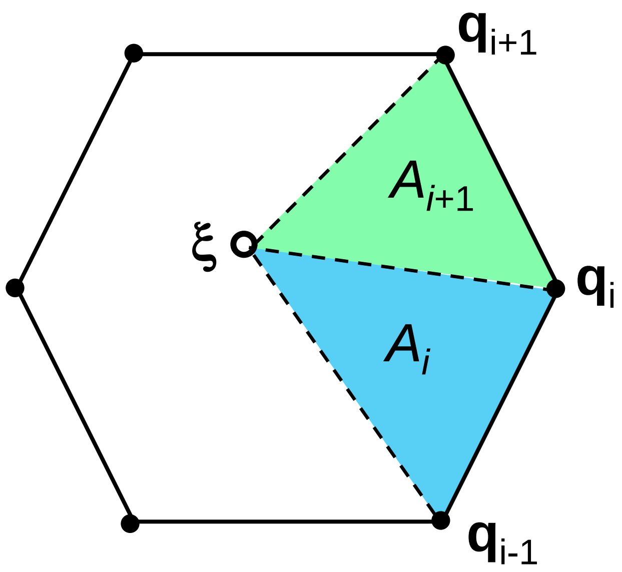
Formulação do elemento poligonal

Funções de forma

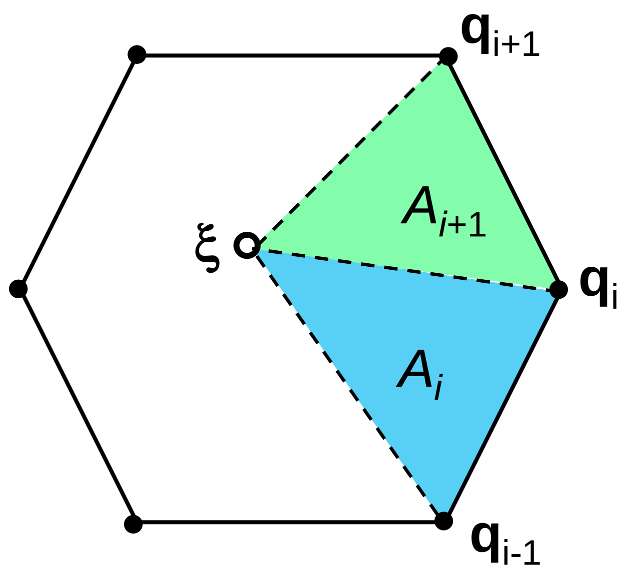
As funções de forma baseadas nas funções de Wachspress são:



$$N_i(\xi) = \frac{\omega_i}{\sum_{k=1}^{n} \omega_k} \qquad \omega_i = \frac{A(\mathbf{q}_{i-1}, \mathbf{q}_i, \mathbf{q}_{i+1})}{A(\mathbf{q}_{i-1}, \mathbf{q}_i, \xi)\, A(\mathbf{q}_i, \mathbf{q}_{i+1}, \xi)}$$

Se o polígono é regular: $A(\mathbf{q}_{i-1}, \mathbf{q}_i, \mathbf{q}_{i+1}) = A_c, \forall i \in [1, n]$

$$\left. \begin{aligned} A_i(\xi) &= A(\mathbf{q}_{i-1}, \mathbf{q}_i, \xi) \\ A_{i+1}(\xi) &= A(\mathbf{q}_i, \mathbf{q}_{i+1}, \xi) \end{aligned} \right\} \quad \omega_i = \frac{1}{A_i(\xi)\, A_{i+1}(\xi)}$$



$$A_i(\xi) = \frac{1}{2}\begin{vmatrix} \xi_1 & \xi_2 & 1 \\ q_{1,i-1} & q_{2,i-1} & 1 \\ q_{1,i} & q_{2,i} & 1 \end{vmatrix}$$

$$A_i(\xi) = \frac{1}{2}\left[q_{2,i-1}\xi_1 + q_{1,i}\xi_2 + q_{1,i-1}q_{2,i} - q_{2,i-1}q_{1,i} - q_{1,i-1}\xi_2 - q_{2,i}\xi_1\right]$$

$$A_i(\xi) = \frac{1}{2}\left[\left(q_{2,i-1} - q_{2,i}\right)\xi_1 + \left(q_{1,i} - q_{1,i-1}\right)\xi_2 + q_{1,i-1}q_{2,i} - q_{2,i-1}q_{1,i}\right]$$

Cálculo das derivadas

$$A_i(\xi) = \frac{1}{2}\left[\left(q_{2,i-1} - q_{2,i}\right)\xi_1 + \left(q_{1,i} - q_{1,i-1}\right)\xi_2 + q_{1,i-1}q_{2,i} - q_{2,i-1}q_{1,i}\right]$$

$$\frac{\partial A_i}{\partial \xi_1} = \frac{1}{2}\left(q_{2,i-1} - q_{2,i}\right) \qquad \frac{\partial A_i}{\partial \xi_2} = \frac{1}{2}\left(q_{1,i} - q_{1,i-1}\right)$$

$$\omega_i = \frac{1}{A_i(\xi)A_{i+1}(\xi)} \Rightarrow \frac{\partial \omega_i}{\partial \xi_k} = -\frac{1}{\left(A_i(\xi)A_{i+1}(\xi)\right)^2}\left(\frac{\partial A_i(\xi)}{\partial \xi_k}A_{i+1}(\xi) + A_i(\xi)\frac{\partial A_{i+1}(\xi)}{\partial \xi_k}\right)$$

$$\frac{\partial \omega_i}{\partial \xi_k} = -\frac{1}{A_i(\xi)A_{i+1}(\xi)}\left(\frac{1}{A_i(\xi)}\frac{\partial A_i(\xi)}{\partial \xi_k} + \frac{1}{A_{i+1}(\xi)}\frac{\partial A_{i+1}(\xi)}{\partial \xi_k}\right)$$

Cálculo das derivadas

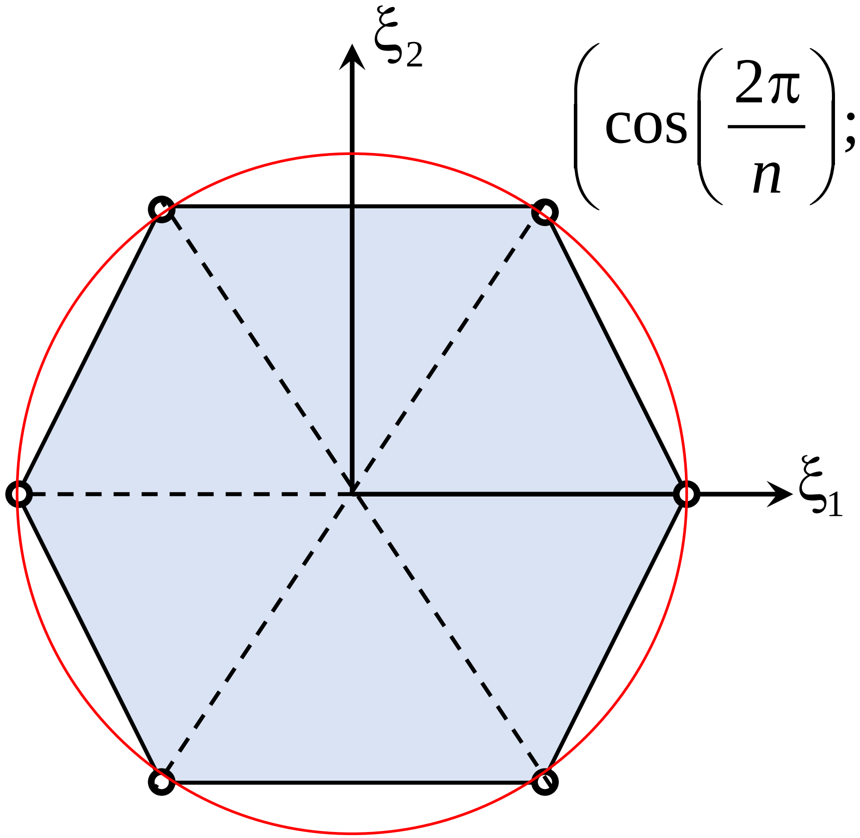
$$\omega_i = \frac{1}{A_i(\xi) A_{i+1}(\xi)} \Rightarrow \frac{\partial \omega_i}{\partial \xi_k} = -\omega_i \left(\frac{1}{A_i(\xi)} \frac{\partial A_i(\xi)}{\partial \xi_k} + \frac{1}{A_{i+1}(\xi)} \frac{\partial A_{i+1}(\xi)}{\partial \xi_k} \right)$$

$$N_i(\xi) = \frac{\omega_i}{\sum_{k=1}^{n} \omega_k} \Rightarrow \frac{\partial N_i}{\partial \xi_k} = \frac{1}{\sum_{k=1}^{n} \omega_k} \frac{\partial \omega_i}{\partial \xi_k} + \omega_i \frac{\partial}{\partial \xi_k} \frac{1}{\sum_{k=1}^{n} \omega_k}$$

$$\frac{\partial N_i}{\partial \xi_k} = \frac{1}{\sum_{j=1}^{n} \omega_j} \frac{\partial \omega_i}{\partial \xi_k} - \omega_i \frac{1}{\left(\sum_{j=1}^{n} \omega_j \right)^2} \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial \xi_k}$$

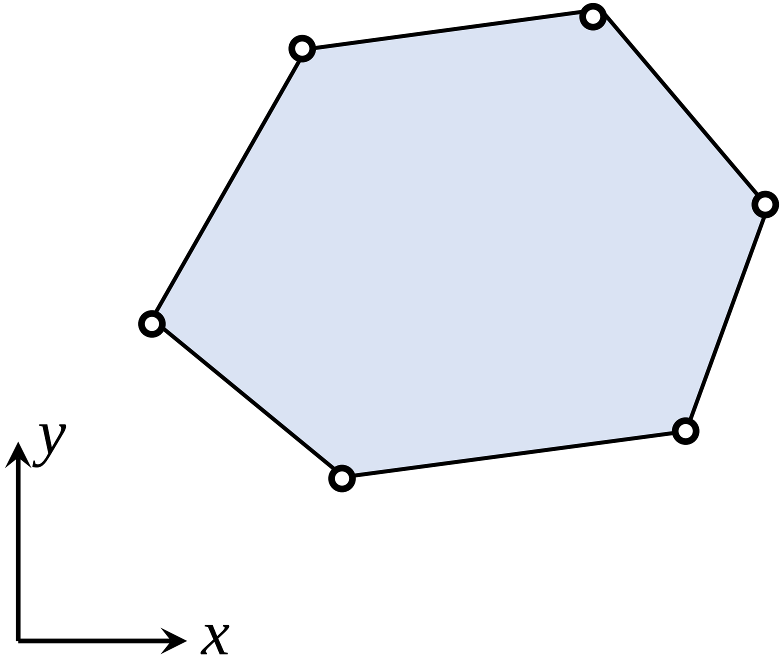
$$\frac{\partial N_i}{\partial \xi_k} = \frac{1}{\sum_{j=1}^{n} \omega_j} \left(\frac{\partial \omega_i}{\partial \xi_k} - N_i \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial \xi_k} \right)$$

Elemento de referência



$$\left(\cos\left(\frac{2\pi}{n}\right);\ \sin\left(\frac{2\pi}{n}\right)\right)$$

Mapeamento isoparamétrico (ϕ)



$$\xi_1 = \cos\left(\frac{2\pi m}{n}\right) \qquad \xi_2 = \sin\left(\frac{2\pi m}{n}\right)$$

Matriz jacobiana do mapeamento

$$\begin{Bmatrix} N_{i,\xi_1} \\ N_{i,\xi_2} \end{Bmatrix} = \begin{bmatrix} x_{,\xi_1} & y_{,\xi_1} \\ x_{,\xi_2} & y_{,\xi_2} \end{bmatrix} \begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix}$$

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} N_{i,\xi_1} \\ N_{i,\xi_2} \end{Bmatrix}$$

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = \boldsymbol{\Gamma} \begin{Bmatrix} N_{i,\xi_1} \\ N_{i,\xi_2} \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} x_{,\xi_1} & y_{,\xi_1} \\ x_{,\xi_2} & y_{,\xi_2} \end{bmatrix} \Rightarrow \mathbf{J} = \begin{bmatrix} \sum \dfrac{\partial N_i}{\partial \xi_1} x_i & \sum \dfrac{\partial N_i}{\partial \xi_1} y_i \\ \sum \dfrac{\partial N_i}{\partial \xi_2} x_i & \sum \dfrac{\partial N_i}{\partial \xi_2} y_i \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi_1} & \dfrac{\partial N_2}{\partial \xi_1} & \cdots & \dfrac{\partial N_n}{\partial \xi_1} \\ \dfrac{\partial N_1}{\partial \xi_2} & \dfrac{\partial N_2}{\partial \xi_2} & \cdots & \dfrac{\partial N_n}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

Deformações

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Gamma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi_1} \\ \dfrac{\partial u}{\partial \xi_2} \\ \dfrac{\partial v}{\partial \xi_1} \\ \dfrac{\partial v}{\partial \xi_2} \end{Bmatrix}$$

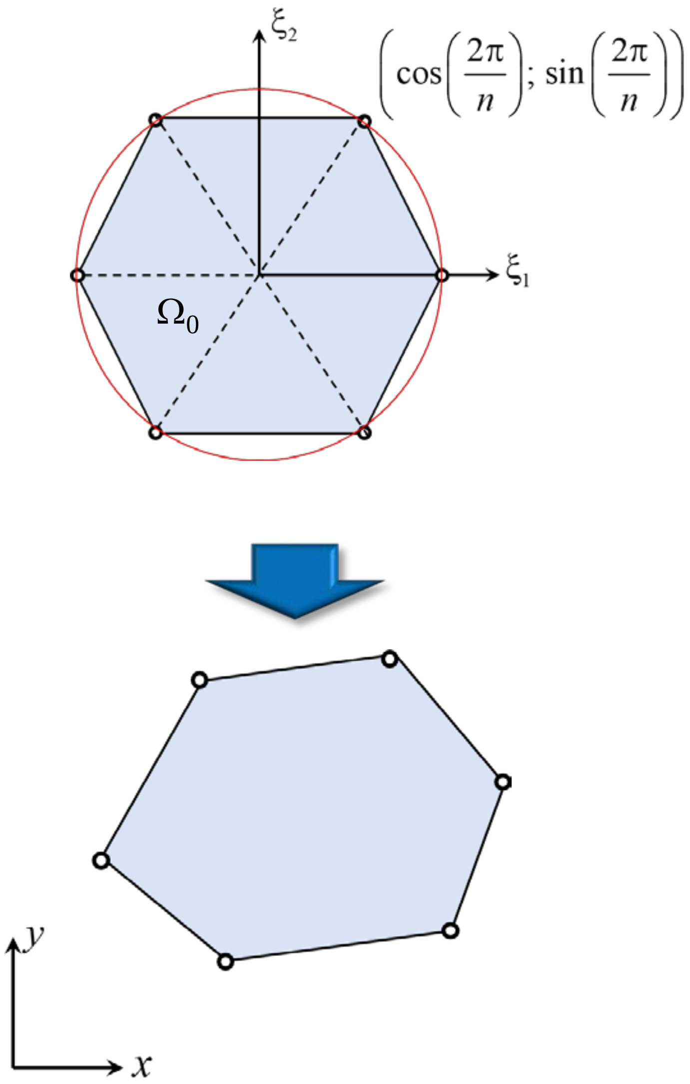
Deformações

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi_1} \\[2ex] \dfrac{\partial u}{\partial \xi_2} \\[2ex] \dfrac{\partial v}{\partial \xi_1} \\[2ex] \dfrac{\partial v}{\partial \xi_2} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & 0 & \cdots & \dfrac{\partial N_n}{\partial \xi_1} & 0 \\[2ex] \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & 0 & \cdots & \dfrac{\partial N_n}{\partial \xi_2} & 0 \\[2ex] 0 & \dfrac{\partial N_1}{\partial \xi_1} & 0 & \dfrac{\partial N_2}{\partial \xi_1} & \cdots & 0 & \dfrac{\partial N_n}{\partial \xi_1} \\[2ex] 0 & \dfrac{\partial N_1}{\partial \xi_2} & 0 & \dfrac{\partial N_2}{\partial \xi_2} & \cdots & 0 & \dfrac{\partial N_n}{\partial \xi_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

Matriz de compatibilidade cinemática

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi_1} & 0 & \frac{\partial N_2}{\partial \xi_1} & 0 & \cdots & \frac{\partial N_n}{\partial \xi_1} & 0 \\ \frac{\partial N_1}{\partial \xi_2} & 0 & \frac{\partial N_2}{\partial \xi_2} & 0 & \cdots & \frac{\partial N_n}{\partial \xi_2} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi_1} & 0 & \frac{\partial N_2}{\partial \xi_1} & \cdots & 0 & \frac{\partial N_n}{\partial \xi_1} \\ 0 & \frac{\partial N_1}{\partial \xi_2} & 0 & \frac{\partial N_2}{\partial \xi_2} & \cdots & 0 & \frac{\partial N_n}{\partial \xi_2} \end{bmatrix}}_{\mathbf{B}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

Integração numérica



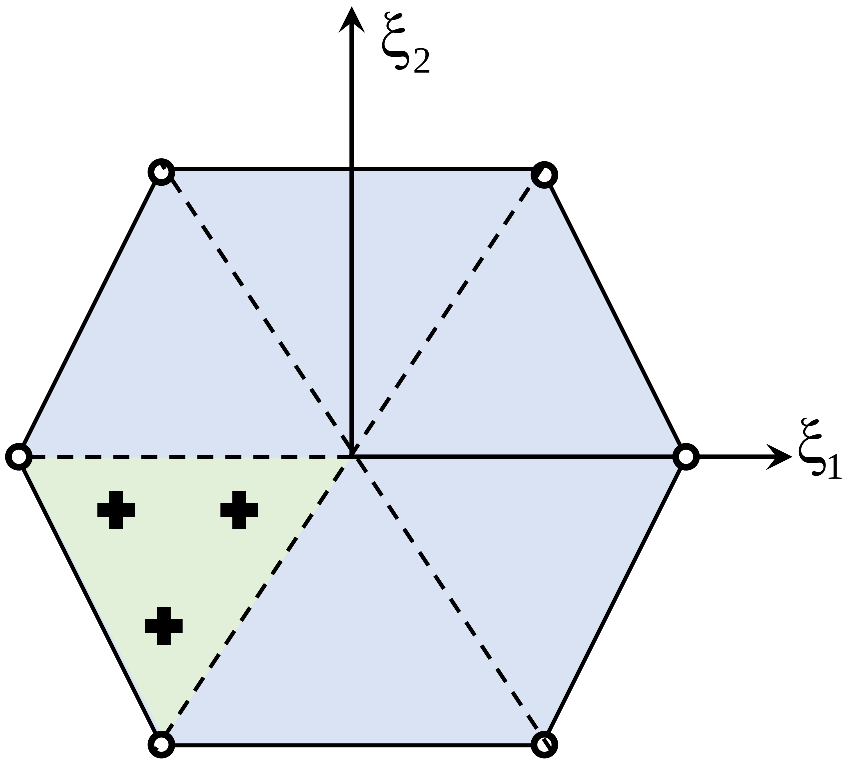
A integração para cálculo da matriz de rigidez do elemento é realizada no domínio do elemento padrão considerando o fator de escala da transformação linear através do determinante do jacobiano do mapeamento:

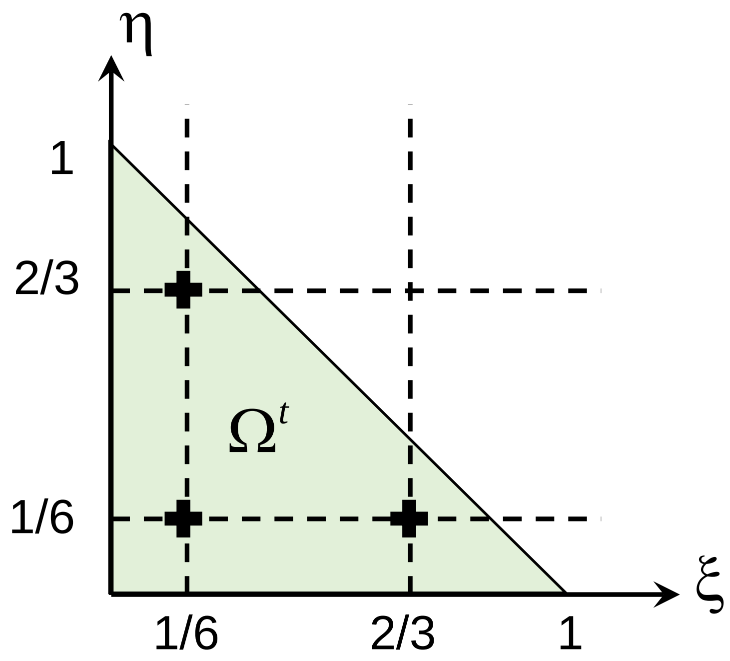
$$\mathbf{k}_e = t \iint_{\Omega_0} \mathbf{B}^T \mathbf{C}\, \mathbf{B} \left|\mathbf{J}\right| d\xi_1 d\xi_2$$

$$f = t\, \mathbf{B}^T \mathbf{C}\, \mathbf{B}$$

Integração numérica

O elemento padrão é particionado em triângulos a fim de que seja possível utilizar uma pontos de Gauss já conhecidos:



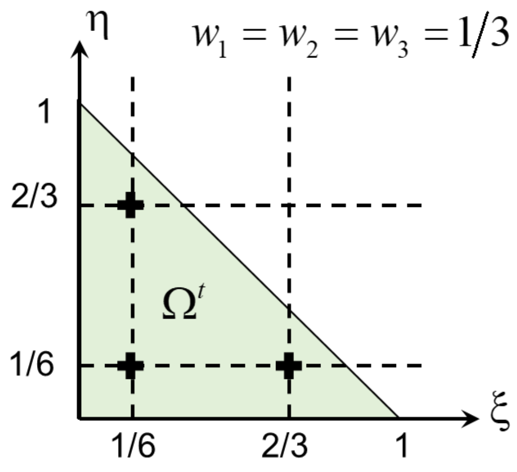


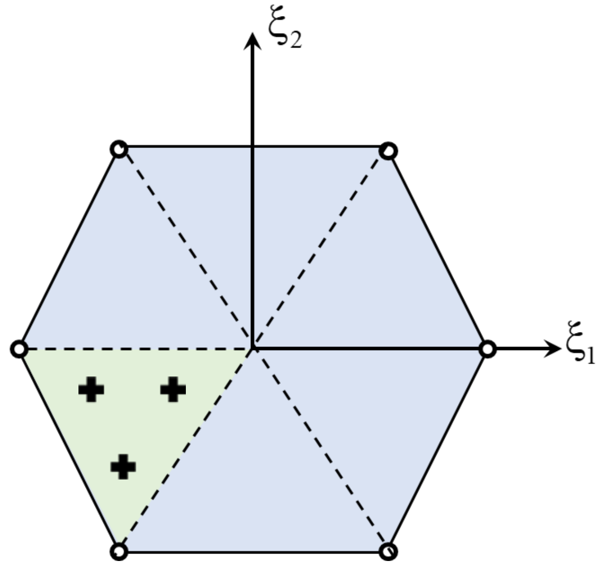
$$\left(\xi_{g1},\eta_{g1}\right)=\left(1/6,1/6\right)\qquad w_1=1/3$$

$$\left(\xi_{g2},\eta_{g2}\right)=\left(1/6,2/3\right)\qquad w_2=1/3$$

$$\left(\xi_{g3},\eta_{g3}\right)=\left(2/3,1/6\right)\qquad w_3=1/3$$

Mapeamento do triângulo





$$\xi_1(\xi,\eta) = \sum_{i=1}^{n} N_i^t(\xi,\eta)\,\xi_1^i$$

$$\xi_2(\xi,\eta) = \sum_{i=1}^{n} N_i^t(\xi,\eta)\,\xi_2^i$$

$$N_1^t(\xi,\eta) = 1 - \xi - \eta$$

$$N_2^t(\xi,\eta) = \xi$$

$$N_1^t(\xi,\eta) = \eta$$

Matriz Jacobiana

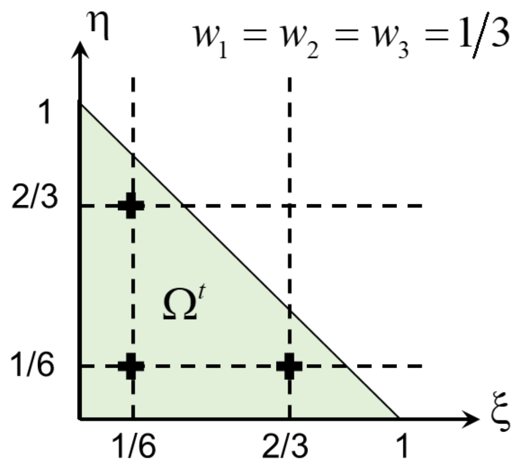
$$\xi_1(\xi,\eta) = \sum_{i=1}^{n} N_i^t(\xi,\eta)\,\xi_1^i \qquad \xi_2(\xi,\eta) = \sum_{i=1}^{n} N_i^t(\xi,\eta)\,\xi_2^i$$

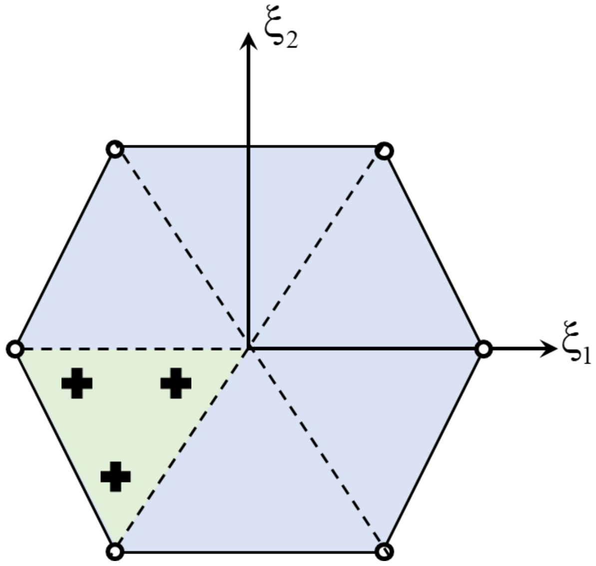
$$N_{i,\xi}^t = N_{i,\xi_1}^t \xi_{1,\xi} + N_{i,\xi_2}^t \xi_{2,\xi}$$
$$N_{i,\eta}^t = N_{i,\xi_1}^t \xi_{1,\eta} + N_{i,\xi_2}^t \xi_{2,\eta}$$

$$\begin{Bmatrix} N_{i,\xi}^t \\ N_{i,\eta}^t \end{Bmatrix} = \underbrace{\begin{bmatrix} \xi_{1,\xi} & \xi_{2,\xi} \\ \xi_{1,\eta} & \xi_{2,\eta} \end{bmatrix}}_{\mathbf{J}^t} \begin{Bmatrix} N_{i,\xi_1}^t \\ N_{i,\xi_2}^t \end{Bmatrix}$$

$$\mathbf{J}^t = \begin{bmatrix} \dfrac{\partial N_1^t}{\partial \xi_1} & \dfrac{\partial N_2^t}{\partial \xi_1} & \dfrac{\partial N_3^t}{\partial \xi_1} \\[2ex] \dfrac{\partial N_1^t}{\partial \xi_2} & \dfrac{\partial N_2^t}{\partial \xi_2} & \dfrac{\partial N_3^t}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & \xi_2^2 \\ \xi_1^3 & \xi_2^3 \end{bmatrix}$$

Integração numérica



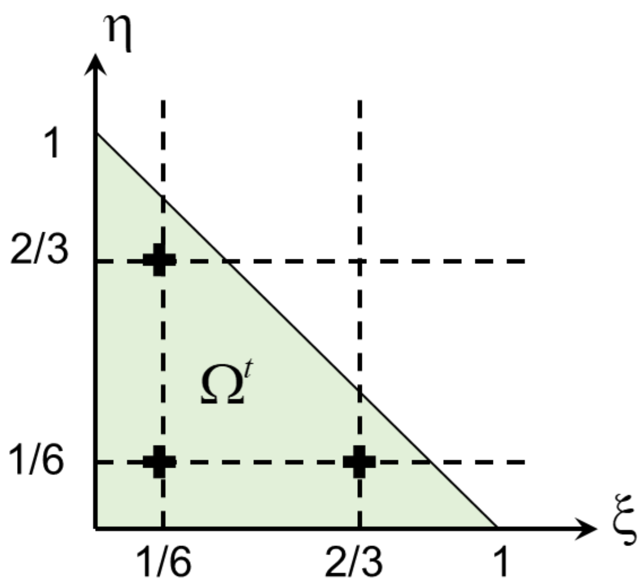


$$\iint_{\Omega_0} f\,|\mathbf{J}|\,d\xi_1 d\xi_2 = \sum_{i=1}^{n} \int_0^1 \int_0^{1-\xi} f\,\left|\mathbf{J}_i^t\right|\,|\mathbf{J}|\,d\xi\, d\eta$$

$$\int_0^1 \int_0^{1-\xi} f\,\left|\mathbf{J}_i^t\right|\,|\mathbf{J}|\,d\xi\, d\eta \approx \frac{1}{2} \sum_{j=1}^{npG} \left|\mathbf{J}_i^t\right|\,|\mathbf{J}|\,w_j\, f\left(\xi_{1j}, \xi_{2j}\right)$$

Pontos de Gauss

$$\int_0^1 \int_0^{1-\xi} f \left|\mathbf{J}_i^t\right| \left|\mathbf{J}\right| d\xi \, d\eta \approx \frac{1}{2} \sum_{j=1}^{npG} \left|\mathbf{J}_i^t\right| \left|\mathbf{J}\right| w_j \, f\left(\xi_{1j}, \xi_{2j}\right)$$



$$\left(\xi_{g1}, \eta_{g1}\right) = \left(1/6, 1/6\right) \qquad w_1 = 1/3$$

$$\left(\xi_{g2}, \eta_{g2}\right) = \left(1/6, 2/3\right) \qquad w_2 = 1/3$$

$$\left(\xi_{g3}, \eta_{g3}\right) = \left(2/3, 1/6\right) \qquad w_3 = 1/3$$

Substitui os pontos de Gauss em:

$$\xi_{1j}\left(\xi_{gj}, \eta_{gj}\right) = \sum_{i=1}^{n} N_i^t\left(\xi_{gj}, \eta_{gj}\right) \xi_1^i \qquad \xi_{2j}\left(\xi_{gj}, \eta_{gj}\right) = \sum_{i=1}^{n} N_i^t\left(\xi_{gj}, \eta_{gj}\right) \xi_2^i$$