Introducció al Mètode dels Elements Finits

Elements de barra

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Teoria de l'elasticitat

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Elements de barra

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Mètode dels Residus Ponderats

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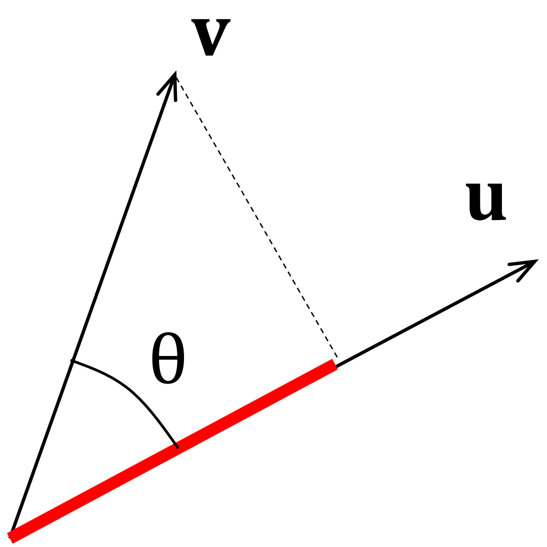


$$\mathbf{u} \cdot \mathbf{v} = u_i v_i = c$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

Projecció d'**u** a **v**, o vice-versa

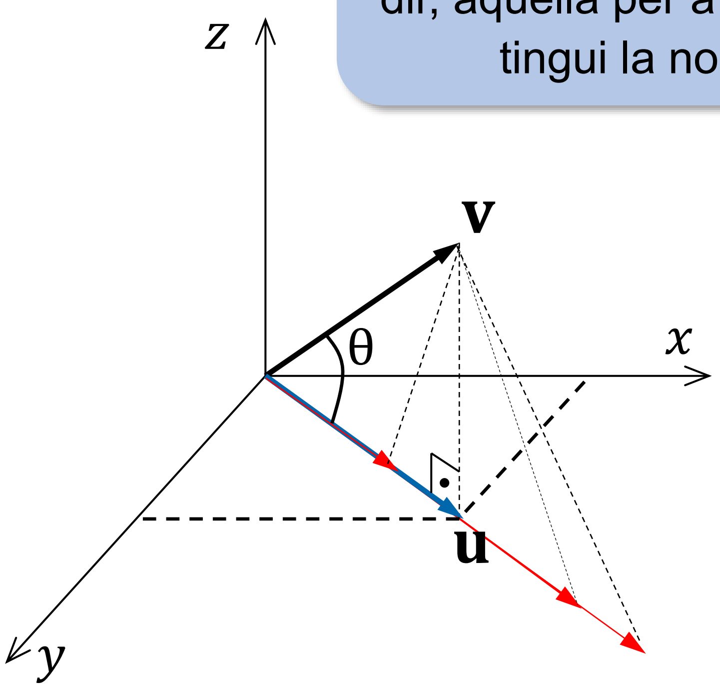
Dos vectors son ortogonals si: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos 0 \rightarrow \mathbf{u} \cdot \mathbf{v} = 0$

Aplicació del concepte d'ortogonalitat

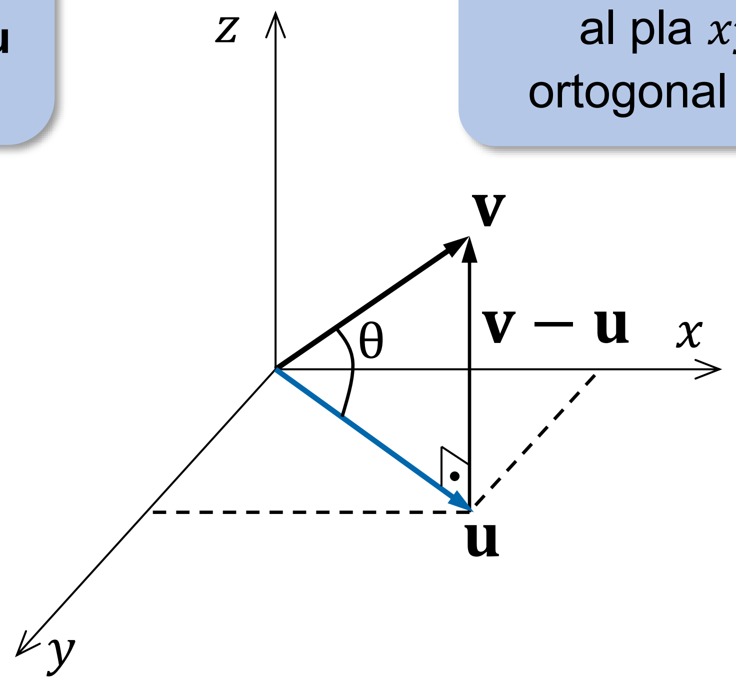
Exemple:

Volem trobar la millor aproximació de **v**(x,y,z) en el pla xy:

La millor aproximació serà aquella que condueixi a l'error més petit, és a dir, aquella per a la qual el vector **v−u** tingui la norma més petita



La norma del vector $(\mathbf{v}-\mathbf{u})$és mínima quan aquest és ortogonal al pla xy, és a dir, quan és ortogonal al vector aproximat $\mathbf{u}$



$$\mathbf{u}\cdot(\mathbf{v}-\mathbf{u}) = 0$$

Ortogonalitat de funcions

Podem generalitzar el concepte de producte intern de vectors per les funcions:

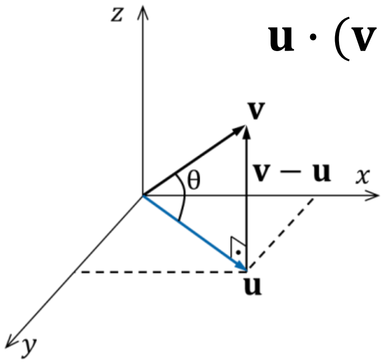
$$\mathbf{u} \cdot \mathbf{v} = \sum u_i v_i \quad \Rightarrow \quad \langle f, g \rangle = \iint_D f(x,y) g(x,y) dxdy$$

Dues funcions *f* i *g* són ortogonals, en un interval, si el producte intern entre elles és nul:

$$\iint_D f(x,y) g(x,y) dxdy = 0$$

Aplicant el concepte d'ortogonalitat de funcions

En el contexte vectorial



$$\mathbf{u} \cdot (\mathbf{v} - \mathbf{u}) = 0$$

En el contexte de funcions:

$$\iint_D f(x,y)g(x,y)dxdy = 0$$

$$\boxed{\iint_D R(a_0, a_1, \dots, a_n, x, y)\phi_j(x,y) = 0}$$

En el Mètode de Galerkin, l'objetiu és minimitzar la funció residu:

Funció d'aproximació

$$u_a = u_0(x,y) + \sum_{j=1}^{N} a_j \phi_j(x,y)$$

Funció residu = Error d'aproximació

$$R(a_0, a_1, \dots, a_n, x, y) = L(u_a)$$

La funció d'aproximació ha de ser ortogonal a l'error

Aplicació a una barra

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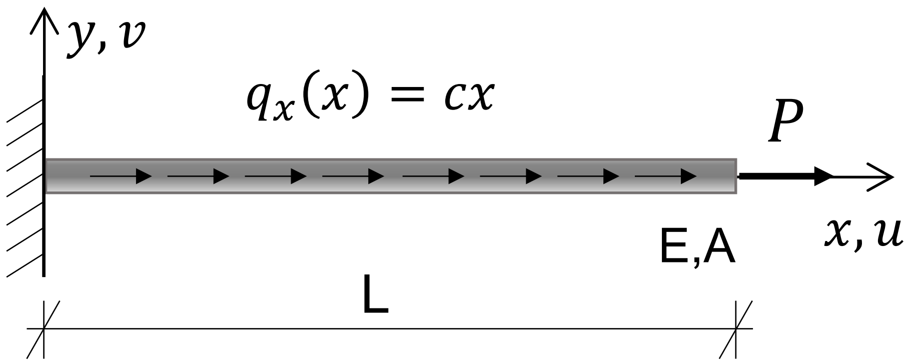
Equilibri en una barra



Equació diferencial:

$$EA\frac{d^2u_0(x)}{dx^2} = -q_x(x)$$

Condicions de contorn:

$$\begin{cases} u_0(0) = 0 \\ N(L) = P \rightarrow \dfrac{du_0(L)}{dx}EA = P \end{cases}$$

Essencial

Natural

Funció d'aproximació:

$$\hat{u}(x) = \sum_{i=0}^{n} a_i x^i$$

Funció resídu:

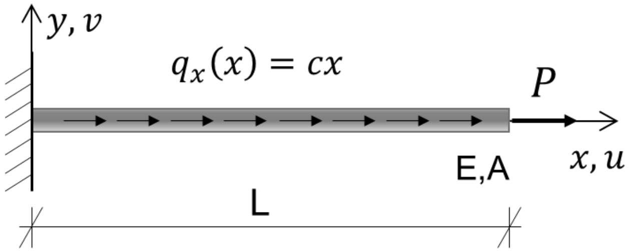
$$R(\hat{u}(x)) = EA\frac{d^2\hat{u}(x)}{dx^2} + q_x(x)$$

Aplicació del MRP:

$$\int_0^L R(\hat{u}(x))W_i\,dx = 0$$

$$\int_0^L \left(EA\frac{d^2\hat{u}(x)}{dx^2} + q_x(x)\right)W_i\,dx = 0$$

Equilibri en una barra – Forma integral



$$\int_0^L \left(EA\frac{d^2\hat{u}(x)}{dx^2} + q_x\right) W_i\, dx = 0 \;\Rightarrow\; \int_0^L \left(EA\frac{d^2\hat{u}(x)}{dx^2}\right) W_i\, dx + \int_0^L (q_x) W_i\, dx = 0$$

Aplicant l'integració per parts: $\int u\, dv = uv - \int v\, du$

$$\int_0^L EA\frac{d^2\hat{u}(x)}{dx^2} W_i\, dx = \left[W_i \frac{d\hat{u}(x)}{dx} EA\right]\Bigg|_0^L - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\, dx$$

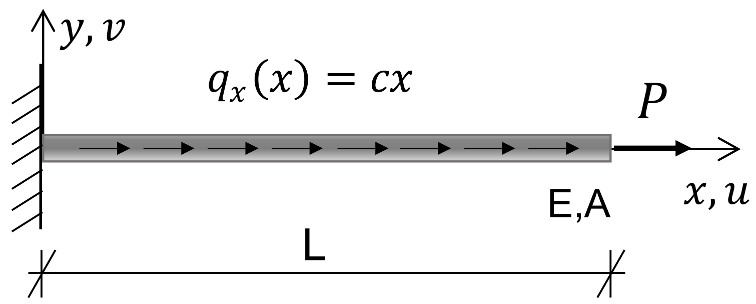
$$\int_0^L \left(EA\frac{d^2\hat{u}(x)}{dx^2} + q_x\right) W_i\, dx = \boxed{\left[W_i \frac{d\hat{u}(x)}{dx} EA\right]\Bigg|_0^L} - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\, dx + \int_0^L (q_x) W_i\, dx$$

$$\int_0^L \left(\frac{d^2\hat{u}(x)}{dx^2} + \frac{c\,x}{EA}\right) W_i\, dx = \boxed{W_i(L)\frac{d\hat{u}(L)}{dx}EA - W_i(0)\frac{d\hat{u}(0)}{dx}EA} - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\, dx + \int_0^L q_x W_i\, dx$$

Condició de contorn natural:

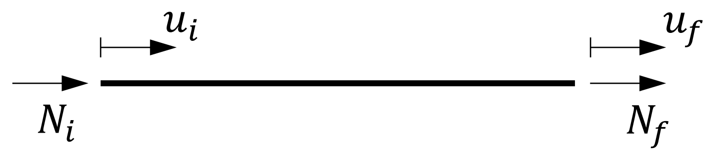
Apareixen naturalment en el procés d'integració per parts

Aplicant les condicions de contorn



$$W_i(L)\boxed{\frac{d\hat{u}(L)}{dx}EA} - W_i(0)\boxed{\frac{d\hat{u}(0)}{dx}EA} - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\,dx + \int_0^L q_x W_i\,dx = 0$$

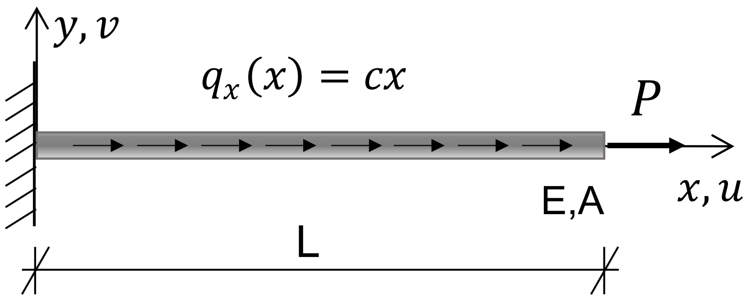
$$\boxed{N(x) = \frac{d\hat{u}(x)}{dx}EA}$$



$$\boxed{W_i(L)\,N_f + W_i(0)\,N_i - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\,dx + \int_0^L q_x W_i\,dx = 0}$$

Aquesta equació significa, físicament, una imposició d'equilibri en forma integral per a un medi unidimensional amb comportament axial.

Exemple



$$\boxed{W_i(L)\,N_f + W_i(0)\,N_i - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\,dx + \int_0^L q_x W_i\,dx = 0}$$

La funció d'aproximació $\hat{u}(x)$ ha de satisfer la condició de contorn essencial del problema:

$$u_a = u_0(x) + \sum_{i=1}^{N} a_i\phi_i(x) \xrightarrow[\phi_i(x) = x^i]{u_0(x) = a_0} \hat{u}(x) = \sum_{i=0}^{n} a_i x^i \xrightarrow[a_0 = 0]{\hat{u}(0) = 0} \hat{u}(x) = \sum_{i=1}^{n} a_i x^i$$

Adoptant *n*=2: $\hat{u}(x) = a_1 x + a_2 x^2$

Pel Mètode dels Residus Ponderats:

$$\int_0^L R\big(\hat{u}(x)\big)W_i\,dx = 0$$

Versió de Galerkin del mètode:

$$W_i(x) = \phi_i(x) = x^i$$

Les funcions de pes són les mateixes que les utilitzades en la funció d'aproximació

Exemple

$$\boxed{W_i(L)\,N_f + W_i(0)\,N_i - \int_0^L EA\frac{d\hat{u}(x)}{dx}\frac{dW_i}{dx}\,dx + \int_0^L q_x W_i\,dx = 0}$$

$$\hat{u}(x) = a_1 x + a_2 x^2 \quad \Rightarrow \quad \begin{array}{l} W_1(x) = x \\ W_2(x) = x^2 \end{array} \quad \Rightarrow \quad \begin{array}{l} {dW_1(x)}/{dx} = 1 \\ {dW_2(x)}/{dx} = 2x \end{array}$$

$i = 1$:
$$L \cdot P - 0 \cdot (a_1 + 2a_2 L) \cdot EA - \int_0^L EA \cdot (a_1 + 2a_2 x) \cdot 1 dx + \int_0^L cx \cdot x\,dx = 0$$

$$L \cdot P - EA \cdot (a_1 L + a_2 L^2) + \frac{c\,L^3}{3} = 0$$

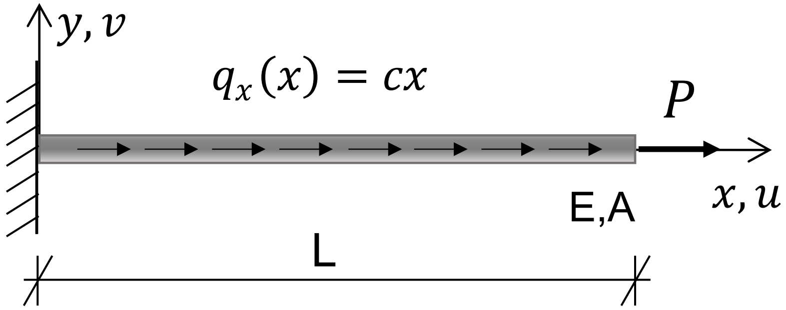
$i = 2$:
$$L^2 \cdot P - 0 \cdot (a_1 + 2a_2 L) \cdot EA - \int_0^L EA \cdot (a_1 + 2a_2 x) \cdot 2x\,dx + \int_0^L cx \cdot x^2 dx = 0$$

$$L^2 \cdot P - EA \cdot \left(a_1 L^2 + a_2 \frac{4L^3}{3}\right) + \frac{c\,L^4}{4} = 0$$

Exemple

$$\begin{cases} L\cdot P - EA\cdot(a_1L + a_2L^2) + \dfrac{c\,L^3}{3} = 0 \\[2ex] L^2\cdot P - EA\cdot\left(a_1L^2 + a_2\dfrac{4L^3}{3}\right) + \dfrac{c\,L^4}{4} = 0 \end{cases} \Rightarrow \begin{cases} a_1L + a_2L^2 = \dfrac{PL}{EA} + \dfrac{c\,L^3}{3EA} \\[2ex] a_1L^2 + a_2\dfrac{4L^3}{3} = \dfrac{PL^2}{EA} + \dfrac{c\,L^4}{4EA} \end{cases}$$

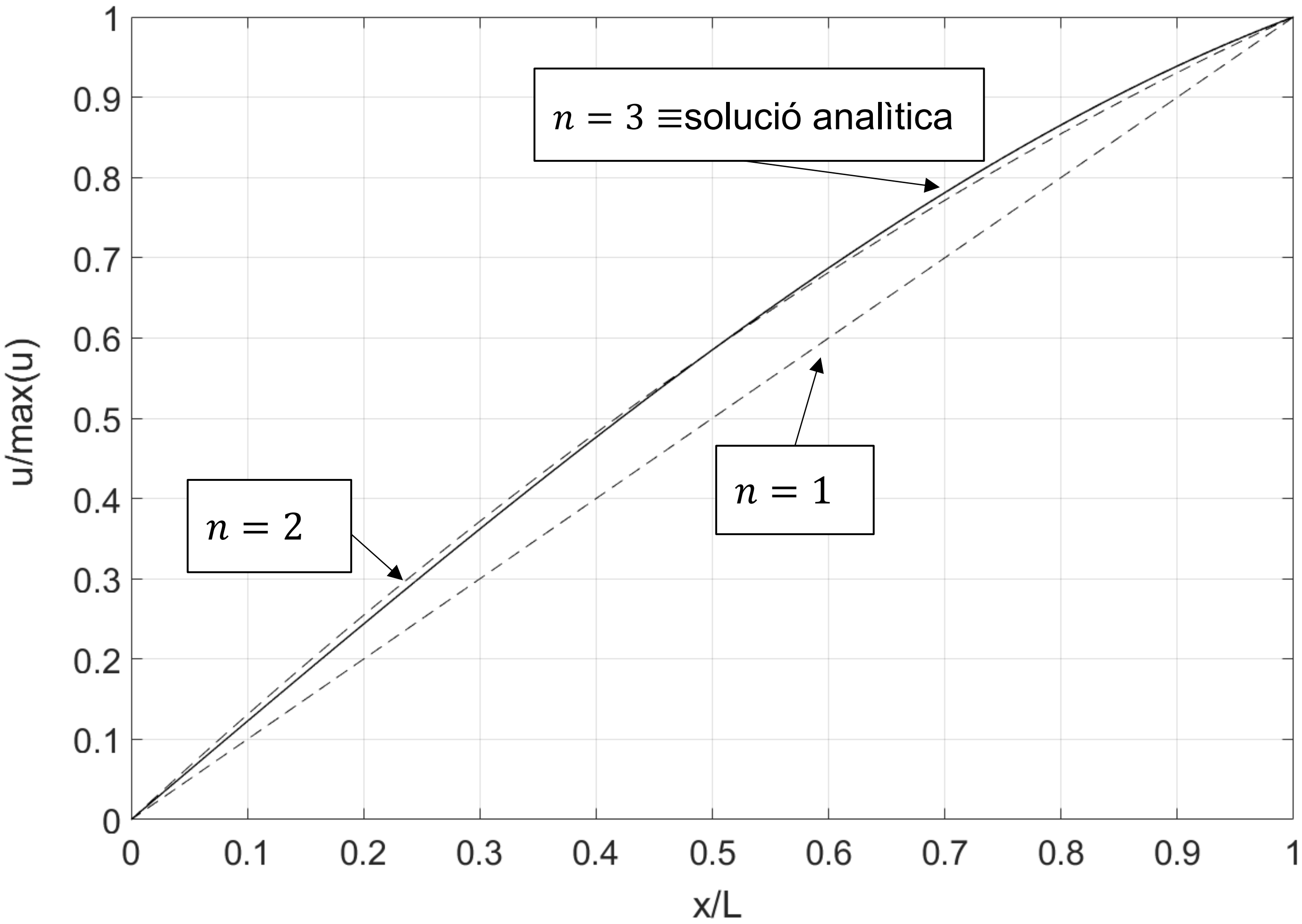
$$\Rightarrow \begin{cases} a_1 + a_2L = \dfrac{P}{EA} + \dfrac{c\,L^2}{3EA} \\[2ex] a_1 + a_2\dfrac{4L}{3} = \dfrac{P}{EA} + \dfrac{c\,L^2}{4EA} \end{cases} \Rightarrow \begin{aligned} a_1 &= \dfrac{P}{EA} + \dfrac{7c\,L^2}{12EA} \\[2ex] a_2 &= -\dfrac{c\,L}{4EA} \end{aligned}$$



$$\hat{u}(x) = \left(\frac{P}{EA} + \frac{7c\,L^2}{12EA}\right)x - \frac{c\,L}{4EA}x^2$$

Desplaçament vs. funció d'aproximació

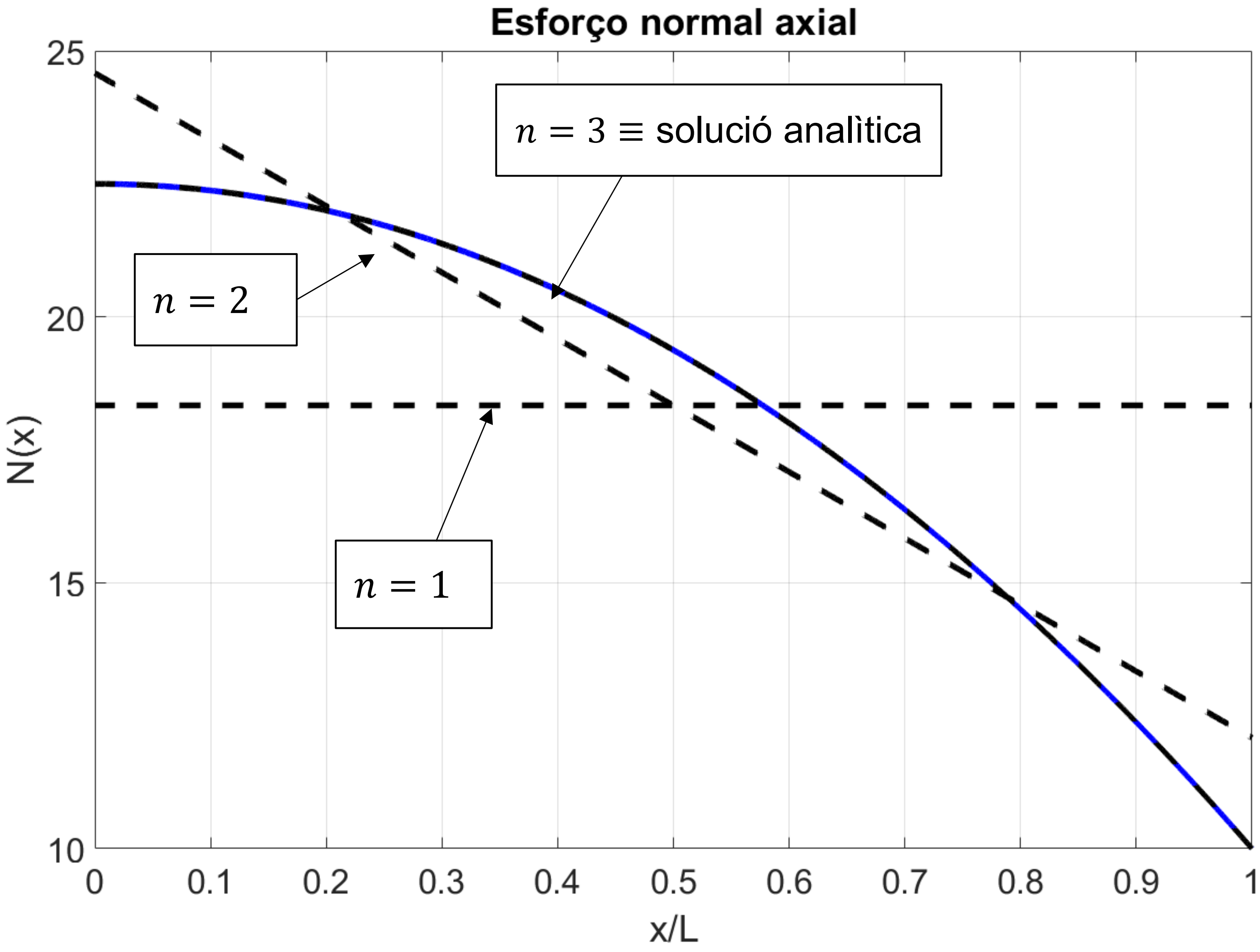
$$\hat{u}(x) = \sum_{i=1}^{n} a_i x^i$$



Esforç normal vs. funció d'aproximació

$$\hat{u}(x) = \sum_{i=1}^{n} a_i x^i$$

O grau de precisão de grandezas que envolvem derivadas da função de aproximação (esforço normal) é menor que o grau de precisão da função de aproximação fundamental;



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Formulació d'un element de barra

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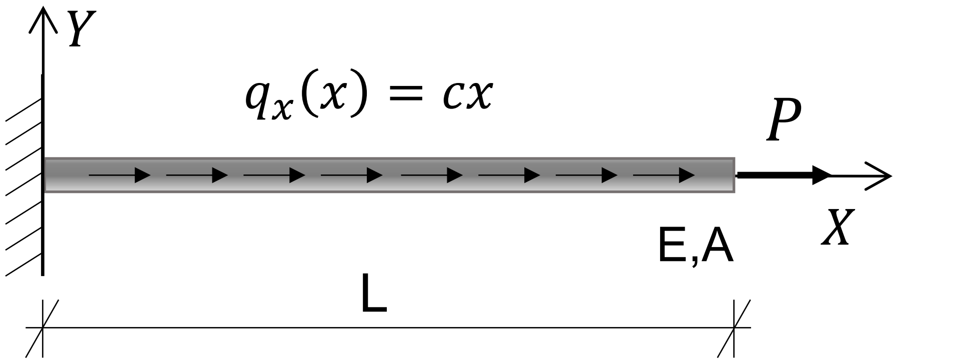
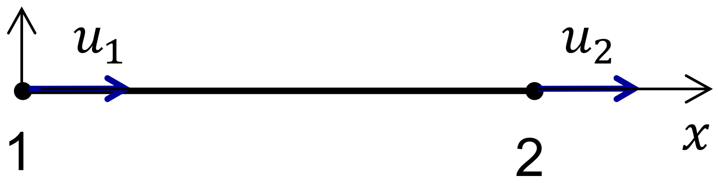
Aproximació del camp de desplaçaments



$$\boxed{\int_0^L EA \frac{du(x)}{dx}\frac{d\delta u(x)}{dx}\,dx = \int_0^L q_x \delta u(x) dx + \delta u(L)\, N_f + \delta u(0)\, N_i}$$



$$\hat{u}(x) = \sum_{i=0}^{n} a_i x^i \xrightarrow{n = 1} \hat{u}(x) = a_0 + a_1 x$$

Sabem:

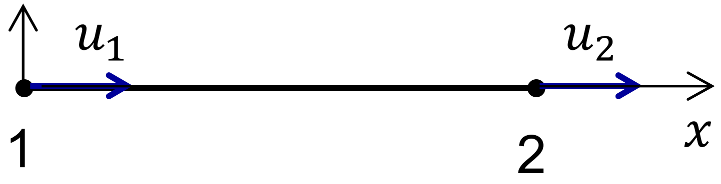
$$\hat{u}(0) = u_1 \qquad \hat{u}(L) = u_2$$

$$\Rightarrow \quad a_0 = u_1 \qquad a_0 + a_1 L = u_2$$

$$\Rightarrow \quad a_0 = u_1 \qquad a_1 = \frac{u_2 - u_1}{L}$$

$$\Rightarrow \quad \hat{u}(x) = u_1 + \frac{u_2 - u_1}{L} x$$

Funció d'interpolació



$$\hat{u}(x) = u_1 + \frac{u_2 - u_1}{L} x$$

$$\hat{u}(x) = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2$$

$$\hat{u}(x) = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\hat{u}(x) = \mathbf{N}(x)\ \mathbf{u}$$

$$\frac{d\hat{u}(x)}{dx} = \varepsilon_x(x) = ?$$

$$\varepsilon_x(x) = \frac{d}{dx}\left(\left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2\right)$$

$$\varepsilon_x(x) = -\frac{1}{L} u_1 + \frac{1}{L} u_2$$

$$\varepsilon_x(x) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \Rightarrow \quad \varepsilon_x(x) = \mathbf{B}(x)\ \mathbf{u}$$

Mètode de Galerkin

$$\int_0^L EA \frac{du(x)}{dx}\frac{d\delta u(x)}{dx}\,dx = \int_0^L q_x \delta u(x) dx + \delta u(L)\, N_f + \delta u(0)\, N_i$$

$$u(x) = \mathbf{N}(x)\,\mathbf{u}$$

$$\varepsilon_x(x) = \mathbf{B}(x)\,\mathbf{u}$$

$\delta u(x) = ?$ Les funcions de pes són les mateixes que les utilitzades en la funció d'aproximació

$$\delta u(x) = \left(1 - \frac{x}{L}\right)\,\delta u_1 + \frac{x}{L}\delta u_2$$

$$\delta u(x) = \mathbf{N}(x)\,\delta\mathbf{u}$$

$$\frac{d\delta u(x)}{dx} = \delta\varepsilon_x(x) = -\frac{1}{L}\,\delta u_1 + \frac{1}{L}\delta u_2 \quad \Rightarrow \quad \delta\varepsilon_x(x) = \mathbf{B}(x)\,\delta\mathbf{u}$$

Mètode de Galerkin

$$\int_0^L EA \frac{d\delta u(x)}{dx}\frac{du(x)}{dx}dx = \int_0^L \delta u(x) q_x dx + \delta u(L)\, N_f + \delta u(0)\, N_i$$

$$u(x) = \mathbf{N}(x)\,\mathbf{u}$$

$$\varepsilon_x(x) = \mathbf{B}(x)\,\mathbf{u}$$

$$\delta u(x) = \mathbf{N}(x)\,\delta\mathbf{u}$$

$$\delta\varepsilon_x(x) = \mathbf{B}(x)\,\delta\mathbf{u}$$

$$\int_0^L EA(\mathbf{B}(x)\delta\mathbf{u})\cdot(\mathbf{B}(x)\,\mathbf{u})dx = \int_0^L (\mathbf{N}(x)\,\delta\mathbf{u})q_x\,dx + (\mathbf{N}(L)\,\delta\mathbf{u})N_f + (\mathbf{N}(0)\,\delta\mathbf{u})N_i$$

$$\int_0^L EA(\mathbf{B}(x)\delta\mathbf{u})^\top(\mathbf{B}(x)\,\mathbf{u})dx = \int_0^L (\mathbf{N}(x)\,\delta\mathbf{u})q_x\,dx + (\mathbf{N}(L)\,\delta\mathbf{u})N_f + (\mathbf{N}(0)\,\delta\mathbf{u})N_i$$

$$\int_0^L \delta\mathbf{u}^\top\, EA\big(\mathbf{B}(x)\big)^\top\mathbf{B}(x)\,\mathbf{u}\,dx = \int_0^L (\mathbf{N}(x)\,\delta\mathbf{u})q_x\,dx + (\mathbf{N}(L)\,\delta\mathbf{u})N_f + (\mathbf{N}(0)\,\delta\mathbf{u})N_i$$

Mètode de Galerkin

$$\int_0^L \delta\mathbf{u}^\top \left(\mathbf{B}(x)\right)^\top E\,A\,\mathbf{B}(x)\,\mathbf{u}\,dx = \int_0^L (\mathbf{N}(x)\,\delta\mathbf{u})q_x\,dx + (\mathbf{N}(L)\,\delta\mathbf{u})N_f + (\mathbf{N}(0)\,\delta\mathbf{u})N_i$$

Treient $\delta\mathbf{u}$ com a factor comú

$$\delta\mathbf{u}^\top \int_0^L \left(\mathbf{B}(x)\right)^\top EA\,\mathbf{B}(x)\,dx\,\mathbf{u} = \int_0^L (\mathbf{N}(x)\,\delta\mathbf{u})q_x\,dx + (\mathbf{N}(L)\,\delta\mathbf{u})N_f + (\mathbf{N}(0)\,\delta\mathbf{u})N_i$$

$$\delta\mathbf{u}^\top \int_0^L \left(\mathbf{B}(x)\right)^\top EA\,\mathbf{B}(x)\,dx\,\mathbf{u} = \delta\mathbf{u}^\top \int_0^L \left(\mathbf{N}(x)\right)^\top q_x\,dx + \delta\mathbf{u}^\top\left(\mathbf{N}(L)\right)^\top N_f + \delta\mathbf{u}^\top\left(\mathbf{N}(0)\right)^\top N_i$$

$$\delta\mathbf{u}^\top \left(\int_0^L \left(\mathbf{B}(x)\right)^\top EA\,\mathbf{B}(x)\,dx\,\mathbf{u} - \int_0^L \left(\mathbf{N}(x)\right)^\top q_x\,dx - \left(\mathbf{N}(L)\right)^\top N_f - \left(\mathbf{N}(0)\right)^\top N_i\right) = 0$$

Mètode de Galerkin

$$\delta\mathbf{u}^{\top}\left(\int_0^L \left(\mathbf{B}(x)\right)^{\top} EA\,\mathbf{B}(x)\,dx\,\mathbf{u} - \int_0^L \left(\mathbf{N}(x)\right)^{\top} q_x\,dx - \left(\mathbf{N}(L)\right)^{\top} N_f - \left(\mathbf{N}(0)\right)^{\top} N_i\right) = 0$$

Per $\delta\mathbf{u} \neq \mathbf{0}$

$$\underbrace{\int_0^L \left(\mathbf{B}(x)\right)^{\top} EA\,\mathbf{B}(x)\,dx}_{\mathbf{k}}\,\mathbf{u} - \underbrace{\int_0^L \left(\mathbf{N}(x)\right)^{\top} q_x\,dx - \left(\mathbf{N}(L)\right)^{\top} N_f - \left(\mathbf{N}(0)\right)^{\top} N_i}_{\mathbf{f}_{\mathrm{e}}} = \mathbf{0}$$

$$\mathbf{k}\,\mathbf{u} = \mathbf{f}_{\mathrm{e}}$$

Element finit de barra

$$\mathbf{N}(x) = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix}$$

$$\mathbf{B}(x) = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

Matriu de rigidesa:

$$\mathbf{k} = \frac{EA}{L}\int_0^L \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} dx \Rightarrow \mathbf{k} = \frac{EA}{L^2}\int_0^L \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \Rightarrow \mathbf{k} = \frac{EA}{L^2}\begin{bmatrix} L & -L \\ -L & L \end{bmatrix} \Rightarrow \mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Vector de forces nodals:

$$\mathbf{f}_\mathrm{e} = \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} q_x\, dx + \begin{bmatrix} 0 \\ 1 \end{bmatrix} N_f + \begin{bmatrix} 1 \\ 0 \end{bmatrix} N_i \Rightarrow \mathbf{f}_\mathrm{e} = \int_0^L \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} q_x\, dx + \begin{bmatrix} N_i \\ N_f \end{bmatrix}$$

Transformació entre sistemes de coordenades

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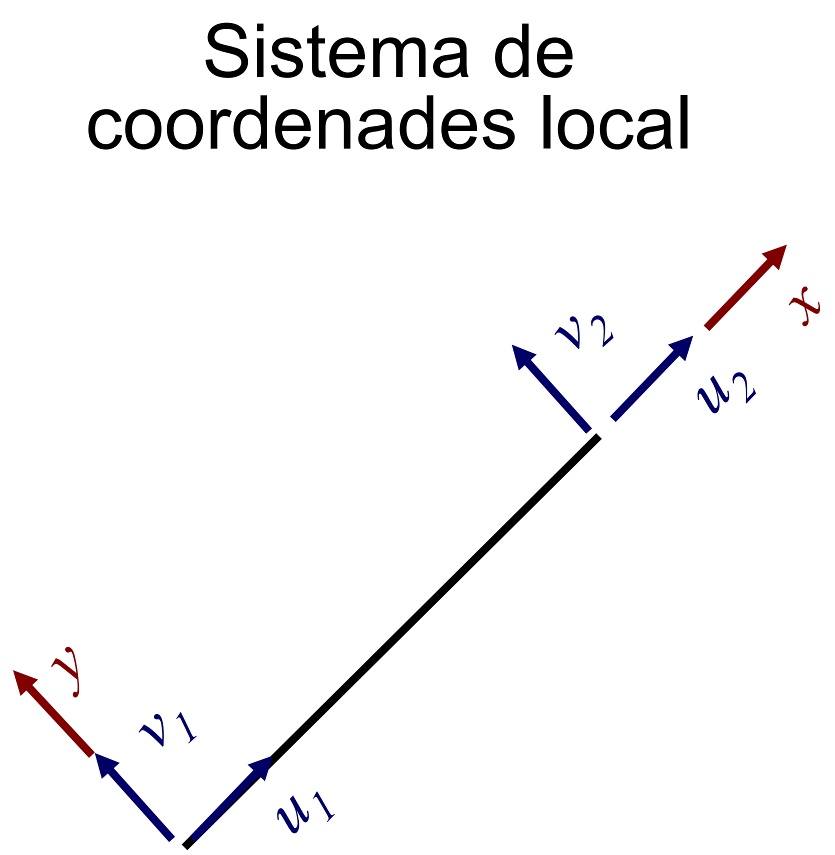
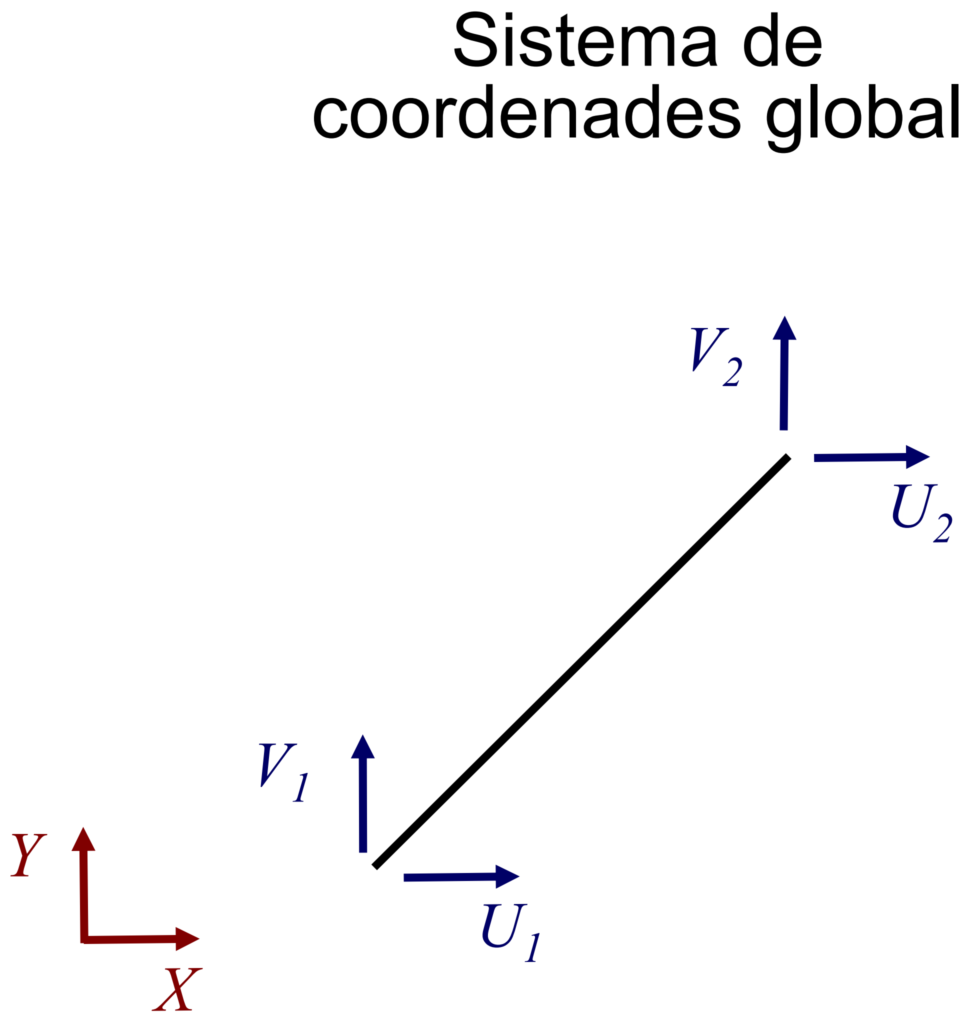
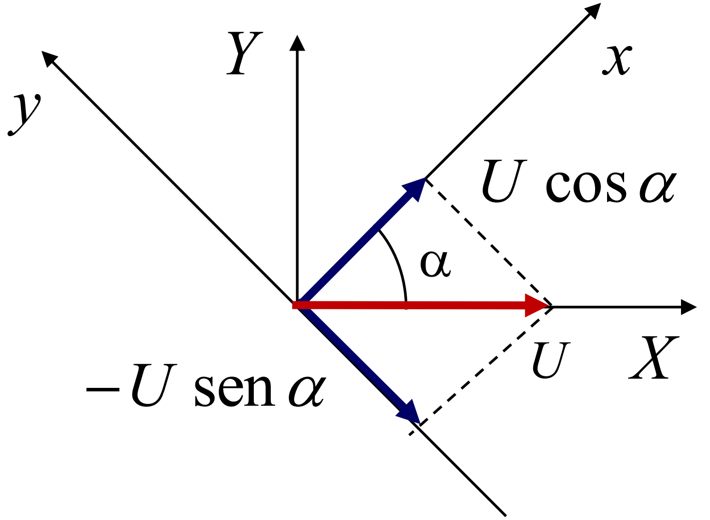
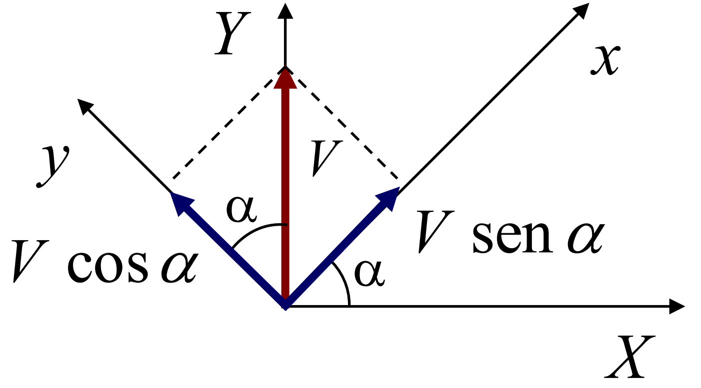
Relació entre sistemes de coordenades





Transformació del sistema global cap al local





Pels desplaçaments

$$u = U\cos\alpha + V\ \mathrm{sen}\,\alpha$$

$$v = -U\ \mathrm{sen}\,\alpha + V\cos\alpha$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \cos\alpha & sen\alpha \\ -sen\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix}$$

sistema local — Matriu de rotació — sistema global

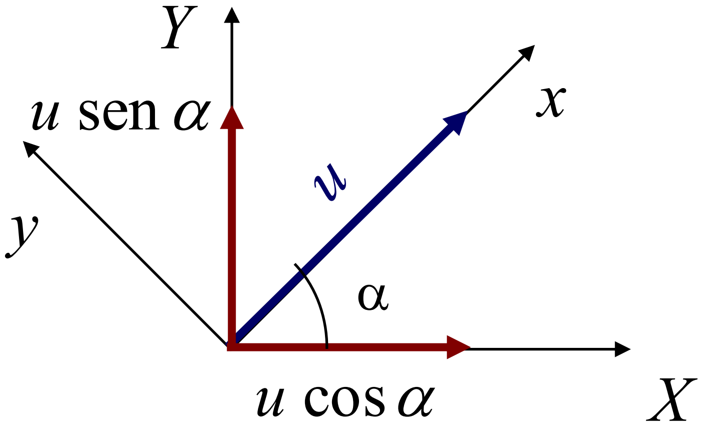
$$\mathbf{u} = \mathbf{R}\ \mathbf{U}$$

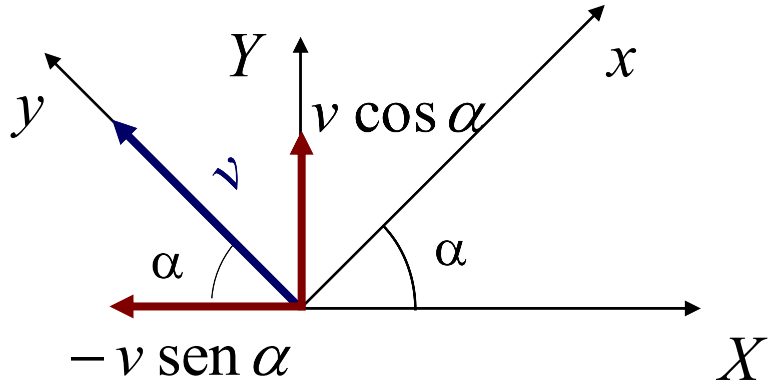
Per les forces:

$$f_x = F_x\cos\alpha + F_y\ sen\alpha$$

$$f_y = -F_x\ sen\alpha + F_y\cos\alpha$$

$$\mathbf{f} = \mathbf{R}\ \mathbf{F}$$





Pels desplaçaments

$$U = u \cos\alpha - v \text{ sen}\, \alpha$$

$$V = u \text{ sen}\, \alpha + v \cos\alpha$$

Per les forces:

$$F_x = f_x \cos\alpha - f_y \, sen\alpha$$

$$F_y = f_x \, sen\alpha + f_y \cos\alpha$$

$$\begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{bmatrix} \cos\alpha & -sen\alpha \\ sen\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\mathbf{U} = \mathbf{R}^T \mathbf{u}$$

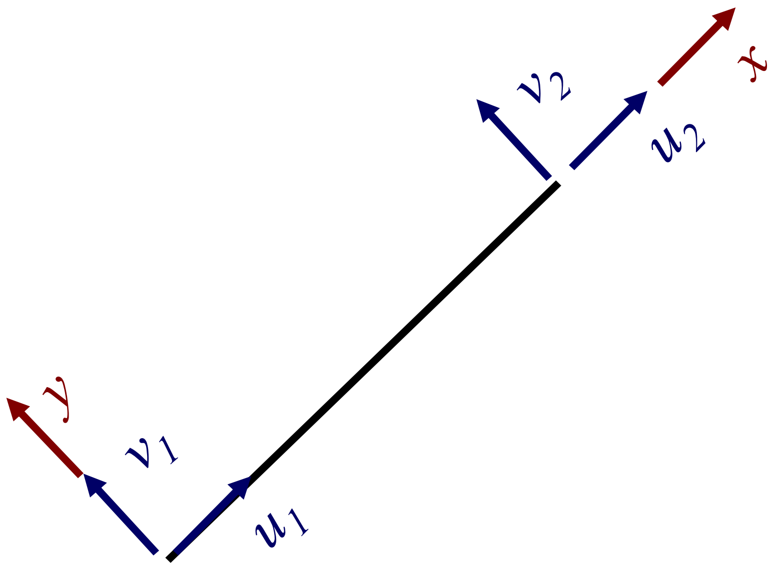
$$\mathbf{F} = \mathbf{R}^T \mathbf{f}$$

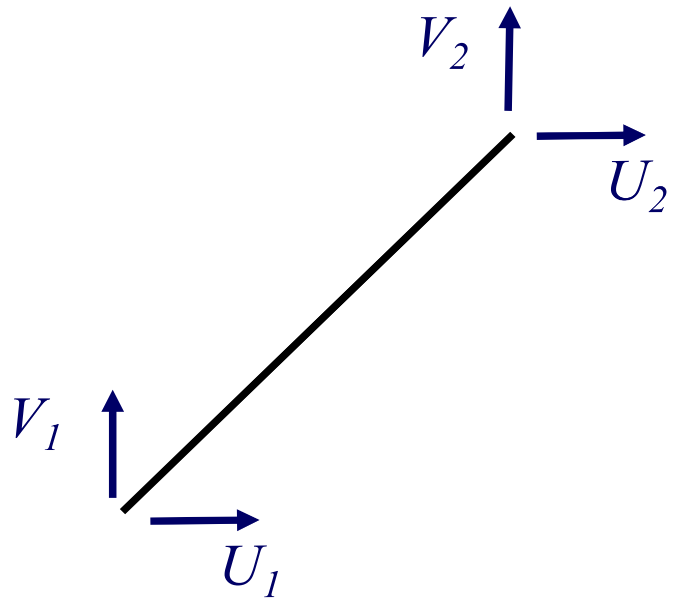
Matriu **R** és ortogonal

Matriu de rotació d'un element de barra amb dos nodes

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & sen\alpha & 0 & 0 \\ -sen\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & sen\alpha \\ 0 & 0 & -sen\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{Bmatrix}$$

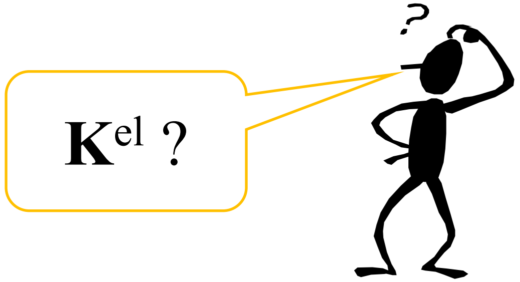
$$\mathbf{u} = \mathbf{R}\ \mathbf{U}$$





Matriu de rigidesa de l'element en el sistema global

$$\mathbf{F}^{el} = \mathbf{K}^{el}\ \mathbf{U}^{el}$$



$$\left\{ \begin{array}{l} \mathbf{u}^{el} = \mathbf{R}^{el}\ \mathbf{U}^{el} \\ \mathbf{f}^{el} = \mathbf{k}^{el}\ \mathbf{u}^{el} \end{array} \right.$$

$$\Rightarrow \quad \mathbf{f}^{el} = \mathbf{k}^{el}\ \mathbf{R}^{el}\ \mathbf{U}^{el}$$

$$\Rightarrow \quad \mathbf{R}^{el\ T}\ \mathbf{f}^{el} = \mathbf{R}^{el\ T}\ \mathbf{k}^{el}\ \mathbf{R}^{el}\ \mathbf{U}^{el}$$

$$\mathbf{R}^{el\ T}\ \mathbf{f}^{el} = \mathbf{R}^{el\ T}\ \mathbf{k}^{el}\ \mathbf{R}^{el}\ \mathbf{U}^{el}$$

$$\mathbf{F}^{el} = \mathbf{R}^{el\ T}\ \mathbf{f}^{el}$$

$$\Rightarrow \quad \mathbf{F}^{el} = \mathbf{R}^{el\ T}\ \mathbf{k}^{el}\ \mathbf{R}^{el}\ \mathbf{U}^{el}$$

$$\mathbf{F}^{el} = \mathbf{K}^{el}\ \mathbf{U}^{el}$$

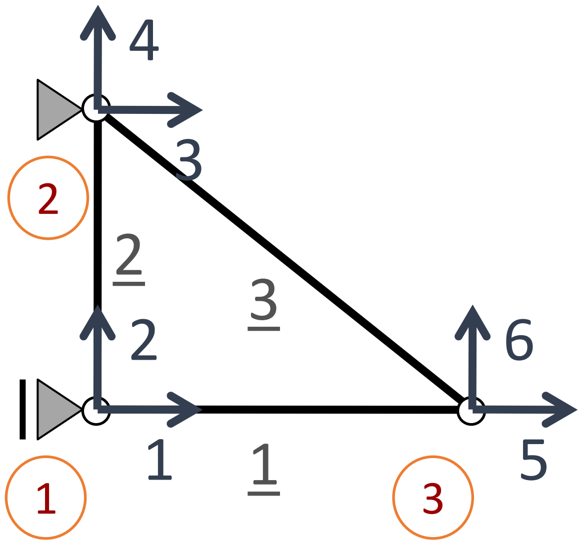
$$\Rightarrow \quad \mathbf{K}^{el} = \mathbf{R}^{el\ T}\ \mathbf{k}^{el}\ \mathbf{R}^{el}$$

Elements de barra

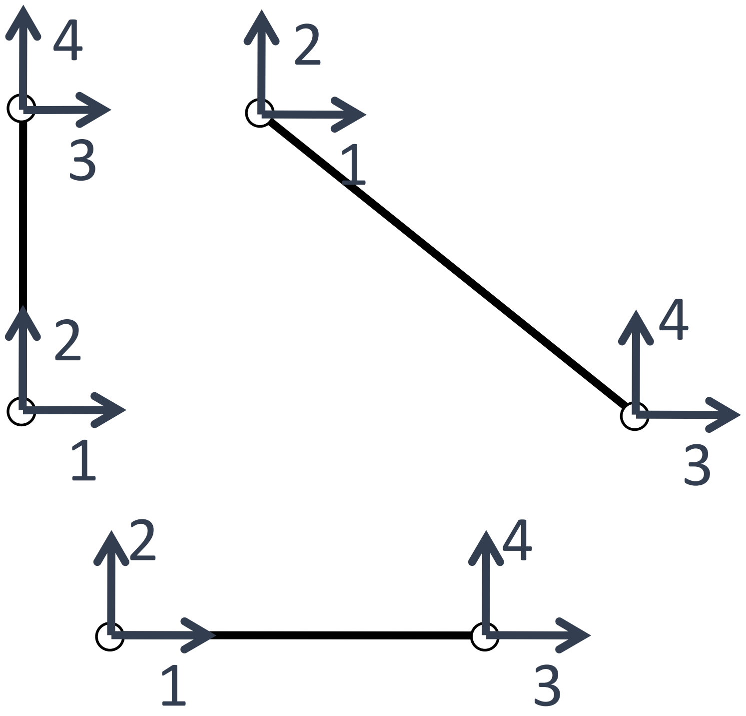
Muntatge de la matriu de rigidesa

D. Borges (UPC)

Matriu de rigidesa de l'estructura

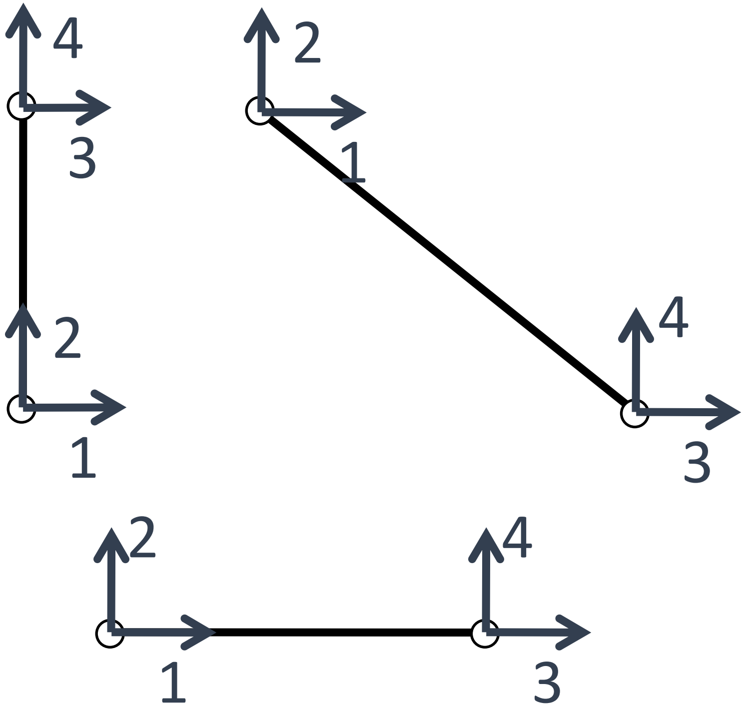


Estructura



Elements

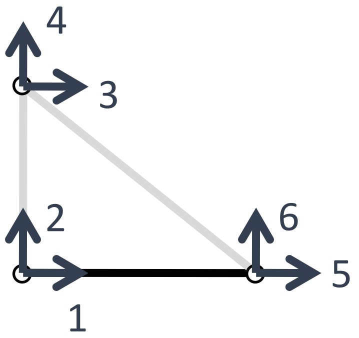
Matriu de rigidesa de l'element (sistema global)



$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & K^e_{11} & K^e_{12} & K^e_{13} & K^e_{14} \\
2 & K^e_{21} & K^e_{22} & K^e_{23} & K^e_{24} \\
3 & K^e_{31} & K^e_{32} & K^e_{33} & K^e_{34} \\
4 & K^e_{41} & K^e_{42} & K^e_{43} & K^e_{44}
\end{array}
$$

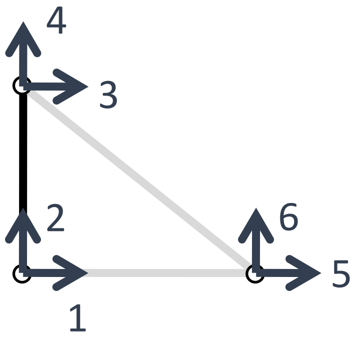
Matriu de rigidesa de l’estructura

$$
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$



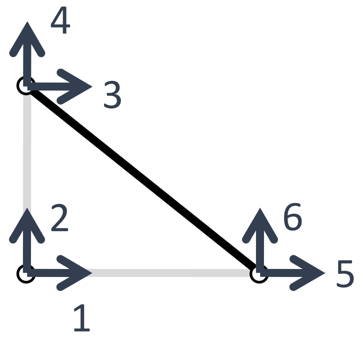
Element 1	1	2	3	4
	☟	☟	☟	☟
Estructura	1	2	5	6

	1	2	3	4	5	6
1	K^{e1}_{11}	K^{e1}_{12}			K^{e1}_{13}	K^{e1}_{14}
2	K^{e1}_{21}	K^{e1}_{22}			K^{e1}_{23}	K^{e1}_{24}
3						
4						
5	K^{e1}_{31}	K^{e1}_{32}			K^{e1}_{33}	K^{e1}_{34}
6	K^{e1}_{41}	K^{e1}_{42}			K^{e1}_{43}	K^{e1}_{44}



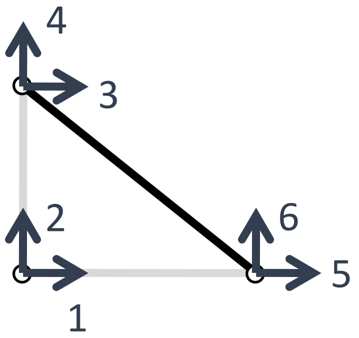
Element 2	1	2	3	4
Estructura	1	2	3	4

	1	2	3	4	5	6
1	$K^{e1}_{11} + K^{e2}_{11}$	$K^{e1}_{12} + K^{e2}_{12}$	K^{e2}_{13}	K^{e2}_{14}	K^{e1}_{13}	K^{e1}_{14}
2	$K^{e1}_{21} + K^{e2}_{21}$	$K^{e1}_{22} + K^{e2}_{22}$	K^{e2}_{23}	K^{e2}_{24}	K^{e1}_{23}	K^{e1}_{24}
3	K^{e2}_{31}	K^{e2}_{32}	K^{e2}_{33}	K^{e2}_{34}		
4	K^{e2}_{41}	K^{e2}_{42}	K^{e2}_{43}	K^{e2}_{44}		
5	K^{e1}_{31}	K^{e1}_{32}			K^{e1}_{33}	K^{e1}_{34}
6	K^{e1}_{41}	K^{e1}_{42}			K^{e1}_{43}	K^{e1}_{44}



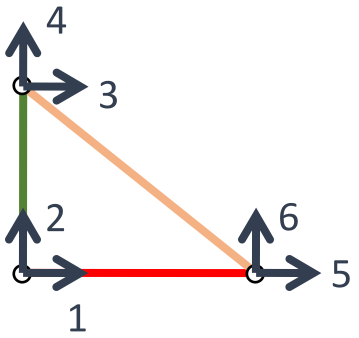
Element 3	1	2	3	4
	☟	☟	☟	☟
Estructura	3	4	5	6

	1	2	3	4	5	6
1	$K^{e1}_{11} + K^{e2}_{11}$	$K^{e1}_{12} + K^{e2}_{12}$	K^{e2}_{13}	K^{e2}_{14}	K^{e1}_{13}	K^{e1}_{14}
2	$K^{e1}_{21} + K^{e2}_{21}$	$K^{e1}_{22} + K^{e2}_{22}$	K^{e2}_{23}	K^{e2}_{24}	K^{e1}_{23}	K^{e1}_{24}
3	K^{e2}_{31}	K^{e2}_{32}	$K^{e2}_{33} + K^{e3}_{11}$	$K^{e2}_{34} + K^{e3}_{12}$	K^{e3}_{13}	K^{e3}_{14}
4	K^{e2}_{41}	K^{e2}_{42}	$K^{e2}_{43} + K^{e3}_{21}$	$K^{e2}_{44} + K^{e3}_{22}$	K^{e3}_{23}	K^{e3}_{24}
5	K^{e1}_{31}	K^{e1}_{32}	K^{e3}_{31}	K^{e3}_{32}	$K^{e1}_{33} + K^{e3}_{33}$	$K^{e1}_{34} + K^{e3}_{34}$
6	K^{e1}_{41}	K^{e1}_{42}	K^{e3}_{41}	K^{e3}_{42}	$K^{e1}_{43} + K^{e3}_{43}$	$K^{e1}_{44} + K^{e3}_{44}$



Element 3	1	2	3	4
Estructura	3	4	5	6

	1	2	3	4	5	6
1	$K^{e1}_{11} + K^{e2}_{11}$	$K^{e1}_{12} + K^{e2}_{12}$	K^{e2}_{13}	K^{e2}_{14}	K^{e1}_{13}	K^{e1}_{14}
2	$K^{e1}_{21} + K^{e2}_{21}$	$K^{e1}_{22} + K^{e2}_{22}$	K^{e2}_{23}	K^{e2}_{24}	K^{e1}_{23}	K^{e1}_{24}
3	K^{e2}_{31}	K^{e2}_{32}	$K^{e2}_{33} + K^{e3}_{11}$	$K^{e2}_{34} + K^{e3}_{12}$	K^{e3}_{13}	K^{e3}_{14}
4	K^{e2}_{41}	K^{e2}_{42}	$K^{e2}_{43} + K^{e3}_{21}$	$K^{e2}_{44} + K^{e3}_{22}$	K^{e3}_{23}	K^{e3}_{24}
5	K^{e1}_{31}	K^{e1}_{32}	K^{e3}_{31}	K^{e3}_{32}	$K^{e1}_{33} + K^{e3}_{33}$	$K^{e1}_{34} + K^{e3}_{34}$
6	K^{e1}_{41}	K^{e1}_{42}	K^{e3}_{41}	K^{e3}_{42}	$K^{e1}_{43} + K^{e3}_{43}$	$K^{e1}_{44} + K^{e3}_{44}$



	1	2	3	4	5	6
1	$K^{e1}_{11} + K^{e2}_{11}$	$K^{e1}_{12} + K^{e2}_{12}$	K^{e2}_{13}	K^{e2}_{14}	K^{e1}_{13}	K^{e1}_{14}
2	$K^{e1}_{21} + K^{e2}_{21}$	$K^{e1}_{22} + K^{e2}_{22}$	K^{e2}_{23}	K^{e2}_{24}	K^{e1}_{23}	K^{e1}_{24}
3	K^{e2}_{31}	K^{e2}_{32}	$K^{e2}_{33} + K^{e3}_{11}$	$K^{e2}_{34} + K^{e3}_{12}$	K^{e3}_{13}	K^{e3}_{14}
4	K^{e2}_{41}	K^{e2}_{42}	$K^{e2}_{43} + K^{e3}_{21}$	$K^{e2}_{44} + K^{e3}_{22}$	K^{e3}_{23}	K^{e3}_{24}
5	K^{e1}_{31}	K^{e1}_{32}	K^{e3}_{31}	K^{e3}_{32}	$K^{e1}_{33} + K^{e3}_{33}$	$K^{e1}_{34} + K^{e3}_{34}$
6	K^{e1}_{41}	K^{e1}_{42}	K^{e3}_{41}	K^{e3}_{42}	$K^{e1}_{43} + K^{e3}_{43}$	$K^{e1}_{44} + K^{e3}_{44}$